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SPARSITY AND SPATIAL LOCALIZATION MEASURES FOR SPATIALLY DISTRIBUTED SYSTEMS*

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Abstract. We consider the class of spatially decaying systems, where the underlying dynamics are spatially decaying and the sensing and controls are spatially distributed. This class of systems arises in various applications where there is a notion of spatial distance with respect to which couplings between the subsystems can be quantified using a class of coupling weights. We exploit the spatial decay property of the underlying dynamics of this class of systems by introducing a class of sparsity and spatial localization measures. We develop a new methodology based on concepts of q-Banach algebras of spatially decaying matrices that enables us to establish a relationship between spatial decay properties of spatially decaying systems and their sparsity and spatial localization features. Moreover, it is shown that the inverse-closedness property of matrix algebras plays a central role in exploiting various structural properties of spatially decaying systems. We characterize conditions for exponential stability of spatially decaying system over q-Banach algebras and prove that the unique solutions of the Lyapunov and Riccati equations over a proper q-Banach algebra also belong to the same q-Banach algebra. It is shown that the quadratically optimal state feedback controllers for spatially decaying systems are sparse and spatially localized in the sense that they have near-optimal sparse information structures.

Key words. distributed control systems, infinite-dimensional systems, optimal control, sparsity, spatial localization, spatially decaying systems

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1. Introduction.

In a number of important applications, centralized implementation of automatic control is practically infeasible due to lack of access to centralized information. This necessitates the design of dynamical networks with sparse and spatially localized information structures [1, 3, 4, 15, 24, 29, 30, 38]. In this paper, we investigate this problem for a class of spatially distributed systems, so-called spatially decaying systems, which are linear systems with off-diagonally decaying state-space matrices. Examples of such systems include linearized and/or spatially discretized models of spatially distributed power networks with sparse interconnection topologies, multiagent systems with nearest-neighbor coupling structures [9], arrays of micromirrors and microcantilevers, and sensor networks. These systems belong to the class of spatio-temporal systems, where all relevant signals are indexed by a spatial coordinate in addition to time [4].

An important class of spatially distributed systems includes spatio-temporal systems. This class of systems can be defined over continuous or (infinite- or finite-

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dimensional) discrete spatial domains. For example, the state-space matrices of spatially invariant systems consist of translation-invariant operators such as partial differential operators with constant coefficients, spatial shift operators, spatial convolution operators, general pseudodifferential operators, and integral operators [17, 45], or a linear combination of such operators [23]. A subclass of spatially invariant systems is considered in [4, 13], where the symmetric spatial invariance properties of this class of systems are exploited and techniques from spatial Fourier transforms are applied to study optimal control of linear spatially invariant systems.

In this paper, we consider spatially distributed systems over (in)finite-dimensional discrete spatial domains, where the dynamics of individual subsystems are heterogeneous, and their spatial interconnection topology does not necessarily exhibit any particular spatial symmetry. Therefore, standard tools such as Fourier analysis cannot be used to analyze this class of systems. The existing traditional methods to study this class of problems are usually based on notions of Banach algebras. One of the earliest works in this area is [8], where the algebraic properties of Riccati equations are studied over C^*-subalgebras of the space of bounded linear operators on some Hilbert space. More recent effort is reported in [10], where the algebraic properties of Riccati equations are investigated over noncommutative involutive Banach algebras, which can be considered as a generalization of earlier results of [8]. Our paper is close in spirit to earlier works [31, 32], where a general approach is proposed based on Banach algebras of spatially decaying matrices to analyze the spatial structure of infinite and finite horizon optimal controllers for spatially distributed systems with arbitrary spatial structures. In [31, 32], it is shown that quadratically optimal controllers inherit spatial decay properties of the underlying dynamics of the systems. A basic fundamental property of a Banach algebra is that it is a Banach space and is locally convex. These properties allow us to apply existing methods in the literature to study the class of spatially distributed systems over Banach algebras; see, for example, [4, 8, 10, 31, 32, 33].

We introduce a new class of spatially decaying systems that are defined over q-Banach algebras endowed with matrix q-norms, where q is an exponent strictly greater than 0 and less than or equal to 1. The class of spatially distributed systems considered in [31, 32] and [10] is a special example of our new class of systems which correspond to exponent $q = 1$. When the exponent q is strictly less than 1, a q-Banach algebra becomes locally nonconvex and is not a Banach space. The main contribution of this paper is the development of a methodology to determine the degrees of sparsity and spatial localization for this class of spatially decaying systems. This is done by establishing a connection between a notion of sparsity and the spatial decay property of the class of spatially decaying systems. The bridge connecting these two fundamental notions is built upon the key idea of asymptotically approximating the space of sparse matrices by inverse-closed q-Banach algebras for sufficiently small values of q.

In section 5, we show that characterization of exponential stability for linear systems over q-Banach algebras differs slightly from the standard characterization, where we explicitly quantify the decay rate of the q-norm of the C_0-semigroup. The main control-theoretic results of this paper are in sections 6 and 7, where we prove that the unique solution of the Lyapunov equation for linear autonomous systems and the Riccati equation resulting from the linear quadratic regulator (LQR) problems over a proper q-Banach algebra also belong to the same q-Banach algebra. The significance of these results is discussed in sections 8.4 and 8.6, and it is shown that the underlying information structure of the LQR controller for a spatially decaying system is inherently spatially localized and each local controller needs to receive state information only from some neighboring subsystems. We also characterize a fundamental limit that explains to what degree a stabilizing controller with spatially decaying structure can be sparsified and spatially localized. Moreover, we argue that an intrinsic tradeoff emerges between a desired degree of sparsification and localization and the global performance loss. A probabilistic method and a computational algorithm are proposed in section 8.7 to analyze and compute near-optimal degrees of sparsity for the Gröchenig–Schur class of spatially decaying matrices.

2. Basic notation. Throughout the paper, the underlying discrete spatial domain of a spatially distributed system is denoted by $\mathbb{G}$. The ℓ^q-measure of a vector $x = [x_i]_{i\in\mathbb{G}}$ is defined by

$$\|x\|_{\ell^q(\mathbb{G})} := \begin{cases} \operatorname{card}\big\{i \in \mathbb{G} \big| x_i \neq 0\big\} & \text{if } q = 0, \\ \Big(\sum_{i\in\mathbb{G}} |x_i|^q\Big)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{i\in\mathbb{G}} |x_i| & \text{if } q = \infty, \end{cases} \tag{2.1}$$

where card is the number of distinct elements in a set. Whenever it is not ambiguous, we use the simplified notation $\|x\|_q$ for the ℓ^q-measure of vector x.

A function $\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ is said to be a *quasi distance* on $\mathbb{G}$ if $\rho(i, j) \geq 0$ for all $i, j \in \mathbb{G}$; $\rho(i, j) = 0$ if and only if $i = j$; and $\rho(i, j) = \rho(j, i)$ for all $i, j \in \mathbb{G}$. A quasi distance is more general than a distance, as the triangle inequality is not required. When $\mathbb{G}$ is the vertex set of an unweighted undirected graph of a sparsely connected spatially distributed system, one may use the shortest distance on graphs to define the quasi distance $\rho(i, j)$ from vertex i to vertex j.

3. Problem statement. Let us consider the class of (in)finite-dimensional spatially distributed linear systems with dynamics

$$\dot{x} = Ax + Bu, \tag{3.1}$$

$$y = Cx + Du, \tag{3.2}$$

where it is assumed that the underlying state space is $\ell^2(\mathbb{G})$, all state-space matrices are constant by time, and all relevant signals of the system are indexed by a spatial coordinate in addition to time. The state, input, and output variables are represented by (in)finite-dimensional vectors $x = [x_i]_{i\in\mathbb{G}}$, $u = [u_i]_{i\in\mathbb{G}}$, and $y = [y_i]_{i\in\mathbb{G}}$, respectively. The LQR problem is defined as the problem of minimizing the following quadratic cost function:

$$J = \int_0^\infty \big(x(t)^* Q x(t) + u(t)^* R u(t)\big)\, dt \tag{3.3}$$

subject to dynamics of linear system (3.1)–(3.2) with initial condition $x(0) = x_0 \in \ell^2(\mathbb{G})$. There is a rich literature that considers this problem on Hilbert spaces (see [14] and references therein) and shows that under some standard assumptions the unique solution to this problem is achieved by a linear state feedback control law $u = Kx$ for which

$$K = -R^{-1}B^*X, \tag{3.4}$$

where X is the unique solution of the following Riccati equation:

$$A^*X + XA + Q - XBR^{-1}B^*X = 0. \tag{3.5}$$

The problem. The goal of this paper is to provide quantitative measures to estimate degrees of sparsity and spatial localization of the LQR feedback control gains (3.4) for the class of spatially distributed systems that are defined over q-Banach algebras.

Our aim is to determine communication requirements in the controller array for the optimal solution of the standard LQR problem for linear systems that are defined over q-Banach algebras. We achieve these goals in several technical steps. First, we develop all the necessary tools to study algebraic and closure properties of the unique solution of Riccati equation (3.5) over q-Banach algebras. Then we apply our results to a class of spatially decaying systems that are defined over the Gröchenig–Schur q-Banach algebras and investigate sparsity and spatial localization features of the corresponding LQR feedback control laws.

4. q-Banach algebras of matrices. We consider a general class of matrix algebras, so-called q-Banach algebras, for $0 < q \leq 1$. The definition of this new class of matrices is motivated by the basic properties of the Gröchenig–Schur class of spatially decaying matrices; see section 8.2 for more details.

DEFINITION 4.1. *For $0 < q \leq 1$, a complex vector space of matrices $\mathcal{A}$ is a q-Banach space equipped with a q-norm $\|\cdot\|_{\mathcal{A}}$ if it is complete with respect to the metric*

$$d_{\mathcal{A}}(A, B) := \|A - B\|_{\mathcal{A}}^q \quad \textit{for } A, B \in \mathcal{A},$$

and the q-norm $\|\cdot\|_{\mathcal{A}}$ satisfies

(i) $\|A\|_{\mathcal{A}} \geq 0$, *and* $\|A\|_{\mathcal{A}} = 0$ *if and only if* $A = 0$*;*
(ii) $\|\alpha A\|_{\mathcal{A}} = |\alpha| \; \|A\|_{\mathcal{A}}$ *for all* $\alpha \in \mathbb{C}$ *and* $A \in \mathcal{A}$*; and*
(iii) $\|A + B\|_{\mathcal{A}}^q \leq \|A\|_{\mathcal{A}}^q + \|B\|_{\mathcal{A}}^q$ *for all* $A, B \in \mathcal{A}$.

A q-Banach space $\mathcal{A}$ with $q = 1$ is a Banach space, while it is a quasi-Banach space for any $0 < q \leq 1$, because

$$\|A + B\|_{\mathcal{A}} \leq (\|A\|_{\mathcal{A}}^q + \|B\|_{\mathcal{A}}^q)^{1/q} \leq 2^{1/q}(\|A\|_{\mathcal{A}} + \|B\|_{\mathcal{A}}) \;\text{ for all } A, B \in \mathcal{A}.$$

One can verify that the series $\sum_{n=1}^{\infty} u_n$ converges in $\mathcal{A}$ if $\sum_{n=1}^{\infty} \|u_n\|_{\mathcal{A}}^q < \infty$.

DEFINITION 4.2. *For the range of exponents $0 < q \leq 1$, a q-Banach space $\mathcal{A}$ equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ is a q-Banach algebra if it contains a unit element I, i.e., $M := \|I\|_{\mathcal{A}} < \infty$, and there exists a constant $K_0 > 0$ such that*

(iv) $\|AB\|_{\mathcal{A}} \leq K_0 \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}}$ *for all* $A, B \in \mathcal{A}$.

One can show that the submultiplicative property for a q-Banach algebra holds by rescaling the q-norm as for $\|\cdot\|_{\mathcal{A}}^{\bullet} = K_0\|\cdot\|_{\mathcal{A}}$,

$$\|AB\|_{\mathcal{A}}^{\bullet} \leq \|A\|_{\mathcal{A}}^{\bullet} \|B\|_{\mathcal{A}}^{\bullet} \quad \text{for all } A, B \in \mathcal{A}.$$

DEFINITION 4.3. *For a given exponent $0 < q \leq 1$ and a Banach algebra $\mathcal{B}$, its q-Banach subalgebra $\mathcal{A}$ is a differential Banach subalgebra of order $\theta \in (0, 1]$ if there exists a constant $D > 0$ such that its q-norm satisfies the differential norm property*

$$\|AB\|_{\mathcal{A}}^q \;\leq\; D \; \|A\|_{\mathcal{A}}^q \; \|B\|_{\mathcal{A}}^q \left(\left(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}} \right)^{q\theta} + \left(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}} \right)^{q\theta} \right) \quad \textit{for all } A, B \in \mathcal{A}.$$

For a Banach subalgebra $\mathcal{A}$ (i.e., when $q = 1$), the above differential norm property with $\theta \in (0, 1]$ has been widely used in operator theory and noncommutative geometry [7, 27, 35] and in solving (non)linear functional equations [40, 43]. The differential norm property with $0 < q < 1$ will play a crucial role in establishing the inverse-closedness property for q-Banach algebra, characterizing exponential stability conditions, and exploiting algebraic properties of the unique solutions of Lyapunov and algebraic Riccati equations over q-Banach algebras; see Theorems 4.6, 5.1, 6.1, and 7.6 for more details. One of the interesting and practical examples of a differential q-Banach algebra is $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \le 1$, which is discussed in detail in section 8.2.

Definition 4.4. *For a given exponent $0 < q \le 1$, a q-Banach algebra $\mathcal{A}$ of matrices equipped with q-norm $\|\cdot\|_{\mathcal{A}}$ is said to be proper if it is a q-Banach algebra according to Definition 4.2 with the following additional properties:*

(P1) *$\mathcal{A}$ is closed under the complex conjugate operation,*

$$\|A^*\|_{\mathcal{A}} = \|A\|_{\mathcal{A}} \quad \text{for all } A \in \mathcal{A}; \tag{4.1}$$

(P2) *$\mathcal{A}$ is a subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ and continuously imbedded with respect to it,*

$$\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \le \|A\|_{\mathcal{A}} \quad \text{for all } A \in \mathcal{A}; \tag{4.2}$$

(P3) *$\mathcal{A}$ is a differential Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ of order $\theta \in (0, 1]$.*

We should mention that under condition (4.2) in property (P2) and using the inequality in Definition 4.3, a differential q-Banach subalgebra $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathbb{G}))$ of order θ is also a differential q-Banach subalgebra $\mathcal{A}$ of order $\theta' \in (0, \theta)$.

Definition 4.5. *A subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ is said to be inverse-closed if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies that $A^{-1} \in \mathcal{A}$.*

The following new result shows that under some mild assumptions q-Banach subalgebras of $\mathcal{B}(\ell^2(\mathbb{G}))$ for $0 < q \le 1$ also inherit the inverse-closedness property from $\mathcal{B}(\ell^2(\mathbb{G}))$.

Theorem 4.6. *For a given exponent $0 < q \le 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Then $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{G}))$.*

Our proof of Theorem 4.6 in Appendix 10.5 is constructive and provides explicit estimates for the q-norm of the inverse matrix A^{-1}. The last missing tool in our toolbox is to establish a relationship between the spectrum of a matrix with respect to $\mathcal{B}(\ell^2(\mathbb{G}))$ and its proper q-Banach subalgebras.

Theorem 4.7. *For a given exponent $0 < q \le 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Then*

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}(\ell^2(\mathbb{G}))}(A) \quad \text{for all } A \in \mathcal{A}, \tag{4.3}$$

where the complement of the spectral set of $A \in \mathcal{A}$ is defined by

$$\mathbb{C} \backslash \sigma_{\mathcal{A}}(A) := \big\{\lambda \in \mathbb{C} \;\big|\; \lambda I - A \text{ has bounded inverse in } \mathcal{A}\big\}.$$

This spectral-invariance property plays an important role in studying sparsity and spatial localization features of spatially decaying systems. In the remaining sections, we provide a solution to our control-theoretic problem stated in section 3.

5. Exponential stability over q-Banach algebras. The optimal solution of the LQR problem in section 3 must be exponentially stabilizing. Thus, the notion of exponential stability for linear systems over q-Banach algebras needs to be revisited. It is said that a matrix A in a (q-)Banach algebra $\mathcal{A}$ is *exponentially stable* if there exist strictly positive constants E and α such that

$$\|e^{tA}\|_{\mathcal{A}} \ \leq \ E\, e^{-\alpha t} \quad \text{for all } t \geq 0. \tag{5.1}$$

The exponential stability of matrix A in a proper q-Banach subalgebra $\mathcal{A}$ of $\mathcal{B}(\ell^2(\mathbb{G}))$ is equivalent to its exponential stability in $\mathcal{B}(\ell^2(\mathbb{G}))$. The necessity follows from the fact that $\mathcal{A}$ is continuously imbedded in $\mathcal{B}(\ell^2(\mathbb{G}))$. The sufficiency follows from the following theorem, which expresses this relationship in a more explicit form by characterizing the decay rate of the q-norm of the strongly continuous semigroup of A.

THEOREM 5.1. *Suppose that $0 < q \leq 1$ and $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. If $A \in \mathcal{A}$ is exponentially stable in $\mathcal{B}(\ell^2(\mathbb{G}))$, i.e., there exist some constants $E, \alpha > 0$ such that*

$$\|e^{tA}\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \leq E e^{-\alpha t} \tag{5.2}$$

for all $t \geq 0$, then

$$\|e^{tA}\|_{\mathcal{A}}^q \leq C(t)\ e^{-\alpha q t} \tag{5.3}$$

for all $t \geq 0$, where constants M, D, K_0, θ are defined in Definitions 4.2, 4.3, and 4.4,

$$\begin{aligned} C(t) &= \left(M^q\, \Xi_0\, e^{qE}\right)^{\left(1+2\,t\,\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\right)^{\log_2(2-\theta)}} \left(2DE^{q\theta}\right)^{\omega(t)}, \\ \Xi_0 &= \sum_{n=0}^{\infty} \frac{K_0^{nq}}{(n!)^q} \left(\frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}\right)^{nq}, \end{aligned} \tag{5.4}$$

and

$$\omega(t) = \begin{cases} \frac{1}{1-\theta}\left(2\,t\,\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\right)^{\log_2(2-\theta)} & \text{if } \ \theta \in (0,1), \\ \log_2\left(1 + 2t\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\right) & \text{if } \ \theta = 1. \end{cases}$$

Proof. For all $t \in \left[0,\, \|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}^{-1}\right]$,

$$\|e^{tA}\|_{\mathcal{A}}^q \ \leq \ \sum_{n=0}^{\infty} \left(\frac{t^n}{n!}\right)^q \|A^n \cdot I\|_{\mathcal{A}}^q \ \leq \ M^q \sum_{n=0}^{\infty} \frac{K_0^{nq}}{(n!)^q} \left(\frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}\right)^{nq}. \tag{5.5}$$

The first inequality follows from the triangle inequality for $\|\cdot\|_{\mathcal{A}}^q$ and the Taylor expansion $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$, and the second inequality holds due to the submultiplicative property (iv) in Definition 4.2. From this result, it follows that

$$\|e^{tA}\|_{\mathcal{A}}^q \ \leq \ M^q\ \Xi_0\ e^{\frac{\alpha q}{\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}}\ e^{-\alpha q t} \tag{5.6}$$

for all $t \in \left[0,\, \|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}^{-1}\right]$. From the identity

$$A^{-1} = -\int_0^{\infty} e^{tA} dt \quad \text{in } \ \mathcal{B}(\ell^2(\mathbb{G})),$$

property (P2) in Definition 4.4, and the exponential stability assumption (5.2), one can conclude that $\|A^{-1}\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \le \frac{E}{\alpha}$. Thus,

$$\frac{\alpha}{\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}} \;\le\; \frac{E}{\|A^{-1}\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}} \;\le\; E.$$

This together with (5.6) implies that

$$\|e^{tA}\|_{\mathcal{A}}^q \;\le\; M^q\ \Xi_0\ e^{qE}\ e^{-\alpha q t} \tag{5.7}$$

for all $t \in \big[0, \|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}\big]$. For every $t > \|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}$, let us denote m to be the smallest positive integer such that $2^{-m}t \in (0,\ \|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}]$, or equivalently,

$$t\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\le\; 2^m \;<\; 2t\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}. \tag{5.8}$$

By applying the differential norm property given in Definition 4.3 and the exponential stability assumption (5.2), we get

$$\|e^{tA}\|_{\mathcal{A}}^q \;\le\; 2D\ \|e^{\frac{t}{2}A}\|_{\mathcal{A}}^{q(2-\theta)}\ \|e^{\frac{t}{2}A}\|^{q\theta}_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\le\; 2DE^{q\theta}\ e^{-q\theta\alpha t/2}\ \|e^{\frac{t}{2}A}\|_{\mathcal{A}}^{q(2-\theta)}. \tag{5.9}$$

Then it follows that

$$\|e^{tA}\|_{\mathcal{A}}^q \;\le\; \big(2DE^{q\theta}\big)^{1+(2-\theta)}\ e^{-\frac{\theta}{2}\left(1+\frac{2-\theta}{2}\right)q\alpha t}\|e^{\frac{tA}{4}}\|_{\mathcal{A}}^{q(2-\theta)^2}$$

and multiple substitutions give us

$$\begin{aligned}\|e^{tA}\|_{\mathcal{A}}^q &\le \Big(2DE^{q\theta}\Big)^{\sum_{j=0}^{m-1}(2-\theta)^j}\ e^{-\frac{\theta}{2}\left(\sum_{j=0}^{m-1}\left(\frac{2-\theta}{2}\right)^j\right)q\alpha t}\ \|e^{\frac{tA}{2^m}}\|_{\mathcal{A}}^{q(2-\theta)^m}\\ &\le (M^q\Xi_0 e^{qE})^{(2-\theta)^m}(2DE^{q\theta})^{\sum_{j=0}^{m-1}(2-\theta)^j}e^{-q\alpha t}\end{aligned} \tag{5.10}$$

for all $t \ge \|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}$. In the above inequalities, the first three inequalities follow from applying (5.9) repeatedly, and the last inequality holds by (5.7). This together with (5.8) proves the desired exponential stability property (5.3) in $\mathcal{A}$. ∎

We should emphasize that $C(t)$ in the right-hand side of (5.3) has a polynomial growth for $\theta = 1$ and a subexponential growth for $0 < \theta < 1$. Thus, the right-hand side of (5.3) vanishes exponentially fast as t goes to infinity, which implies that A is exponentially stable in $\mathcal{A}$.

6. Lyapunov equations over q-Banach algebras. In this section, we investigate structural properties of solutions of Lyapunov equations over q-Banach algebras. We recall a well-known result about solving the algebraic Lyapunov equation over $\mathcal{B}(\ell^2(\mathbb{G}))$; for more details, see [5, page 76] and [8, Theorem 1]. Suppose that $Q \in \mathcal{B}(\ell^2(\mathbb{G}))$ is strictly positive on $\ell^2(\mathbb{G})$, and $A \in \mathcal{B}(\ell^2(\mathbb{G}))$ is exponentially stable on $\ell^2(\mathbb{G})$. Then there exists a unique strictly positive solution P in $\mathcal{B}(\ell^2(\mathbb{G}))$ for the Lyapunov equation

$$AP + PA^* + Q = 0, \tag{6.1}$$

which has the following compact form:

$$P \;=\; \int_0^\infty e^{tA^*}\,Q\,e^{tA}\ dt \quad \text{in} \quad \mathcal{B}(\ell^2(\mathbb{G})). \tag{6.2}$$

Now suppose that matrices A and Q in (6.1) belong to a q-Banach algebra $\mathcal{A}$. Our goal is to show that (6.2), the unique solution of (6.1), also belongs to $\mathcal{A}$. Since $\mathcal{A}$ is not a Banach space for $0 < q < 1$, the explicit expression (6.2) cannot be directly applied to tackle this problem. The reason is that continuous integrals, such as (6.2), cannot be defined properly on q-Banach algebras, as it is not a Banach space. In order to overcome this challenge, first we introduce a sequence of matrices $\{P_m\}_{m\geq 0}$ using the following iterative process:

$$P_m \;=\; e^{A^*\,\|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}}\; P_{m-1}\; e^{A\,\|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}} \;+\; P_0 \tag{6.3}$$

for $m \geq 1$, with initial value

$$P_0 \;=\; \int_0^{\|A\|^{-1}_{\mathcal{B}(\ell^2)}} e^{tA^*} Q e^{tA}\, dt \quad \text{in} \quad \mathcal{B}(\ell^2(\mathbb{G})). \tag{6.4}$$

Then we apply the result of Theorem 5.1 to prove the convergence of the proposed iterative procedure in the q-Banach algebra. It can be shown that $P_m \in \mathcal{A}$ for all $m \geq 0$ and that the sequence $P_m, m \geq 0$, converges in $\mathcal{A}$. In fact, the limit of the sequence $\{P_m\}_{m\geq 0}$, as m tends to infinity, converges to the unique solution of (6.1).

Theorem 6.1. *For $0 < q \leq 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Assume that $Q \in \mathcal{A}$ is strictly positive on $\ell^2(\mathbb{G})$ and that $A \in \mathcal{A}$ is exponentially stable on $\ell^2(\mathbb{G})$. Then the unique strictly positive solution of the Lyapunov equation* (6.1) *belongs to $\mathcal{A}$. Moreover, we have the following upper bound estimate:*

$$\|P\|^q_{\mathcal{A}} \;\leq\; \Xi_0^2\, \Pi_0\; \frac{\|Q\|^q_{\mathcal{A}}}{\|A\|^q_{\mathcal{B}(\ell^2(\mathbb{G}))}}, \tag{6.5}$$

where

$$\Pi_0 = 1 + K_0^{2q} \sum_{k=1}^{\infty} \left(M^{2q}\Xi_0^2 e^{2qE}\right)^{2(2k+1)^{\log_2(2-\theta)}} \left(2DE^{q\theta}\right)^{2\omega_k} \exp\left(-\frac{2\alpha q k}{\|A\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}\right),$$

in which constants E, α are defined in (5.2) *and M, D, K_0, θ in Definitions* 4.2–4.4*, and*

$$\omega_k \;=\; \begin{cases} (1-\theta)^{-1}(2k)^{\log_2(2-\theta)} & \text{if } \theta \in (0,1), \\ \log_2(1+2k) & \text{if } \theta = 1. \end{cases}$$

Proof. For an exponentially stable matrix $A \in \mathcal{A}$, let us consider P_0 given by (6.4). It is straightforward to show that

$$P_0 \;=\; \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\|A\|^{-m-n-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}}{(m+n+1)\,m!\,n!}\; (A^*)^m\, Q\, A^n, \tag{6.6}$$

and its q-norm can be bounded from above as follows:

$$\begin{aligned}
\|P_0\|^q_{\mathcal{A}} &\leq \sum_{m,n=0}^{\infty} \left(\frac{\|A\|^{-m-n-1}_{\mathcal{B}(\ell^2)}}{(m+n+1)\,m!\,n!}\right)^q \left\|(A^*)^m\, Q\, A^n\right\|^q_{\mathcal{A}} \\
&\leq \frac{1}{\|A\|^q_{\mathcal{B}(\ell^2)}} \sum_{m,n=0}^{\infty} \left(\frac{\|A\|^{-m-n}_{\mathcal{B}(\ell^2)}}{m!\,n!}\right)^q K_0^{q(m+n)} \|A^*\|^{qm}_{\mathcal{A}} \|A\|^{qn}_{\mathcal{A}} \|Q\|^q_{\mathcal{A}} \\
&= \frac{\|Q\|^q_{\mathcal{A}}}{\|A\|^q_{\mathcal{B}(\ell^2)}}\; \Xi_0^2 < \infty.
\end{aligned} \tag{6.7}$$

This means that $P_0 \in \mathcal{A}$. Since $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ by (4.2), P_0 is a bounded operator on $\ell^2(\mathbb{G})$. From (6.4), it is evident that P_0 is self-adjoint and strictly positive. Now let us consider the iterative process (6.3) for all $m \geq 1$ with initial value P_0. By induction and direct calculations, we get

$$P_m = \sum_{k=0}^{m} e^{kA^* \|A\|^{-1}_{\mathcal{B}(\ell^2)}} P_0 \, e^{kA\|A\|^{-1}_{\mathcal{B}(\ell^2)}} \tag{6.8}$$

and

$$P_m = \int_0^{(m+1)\|A\|^{-1}_{\mathcal{B}(\ell^2)}} e^{tA^*} Q \, e^{tA} \, dt \quad \text{in} \quad \mathcal{B}(\ell^2(\mathbb{G})) \tag{6.9}$$

for all $m \geq 0$. In the next step, we show that this sequence converges in the q-Banach algebra. From (6.8) and the result of Theorem 5.1, we obtain

$$\begin{aligned}\left\|P_{m+1} - P_m\right\|_{\mathcal{A}}^q \leq & \, (K_0^2 \|P_0\|_{\mathcal{A}})^q (M^q \Xi_0 e^{2qE})^{2(2m+1)^{\log_2(2-\theta)}} \\ & \times \left(2DE^{q\theta}\right)^{2\omega_m} e^{-2q\alpha m \|A\|^{-1}_{\mathcal{B}(\ell^2(\mathbb{G}))}}, \quad m \geq 1.\end{aligned}$$

This implies that $P_m \in \mathcal{A}$ for all $m \geq 0$. Since the right-hand side of the above inequality decays exponentially fast as m tends to infinity, we conclude that the sequence $\{P_m\}_{m\geq 0}$ converges in $\mathcal{A}$. Let us denote the limit of this sequence by P_∞, i.e.,

$$P_\infty = \lim_{m\to\infty} P_m \in \mathcal{A}.$$

The exponential stability of A in $\mathcal{B}(\ell^2(\mathbb{G}))$ implies that the series P_m for all $m \geq 1$ converges to $\int_0^\infty e^{tA^*} Q e^{tA} dt$ in $\mathcal{B}(\ell^2(\mathbb{G}))$. Thus,

$$\int_0^\infty e^{tA^*} Q e^{tA} dt = P_\infty \in \mathcal{A},$$

and the unique solution of the Lyapunov equations (6.1) and (6.2) belongs to $\mathcal{A}$. □

The upper bound in (6.5) gives an estimate of the degree to which the unique solution of the Lyapunov equation (6.1) can be sparsified and spatially localized. The result of this theorem can be particularly used to study the problem of disturbance propagation in spatially decaying systems. It turns out that this problem boils down to evaluation of the corresponding controllability Gramian, which is the solution of a Lyapunov equation. Some recent works study this problem for the class of spatially invariant consensus networks over tori [3] and for consensus networks over arbitrary graphs [39].

Remark 6.2. We should highlight that in this paper we do not define general integrals over q-Banach algebras. The key point of our proof is that we can define the very special integral (6.2) over $\mathcal{A}$ when $A \in \mathcal{A}$ has its spectrum on the closed domain contained in the left-half complex plane. We use a specific definition of integration given by (6.4) and (6.9) over $\mathcal{B}(\ell^2(\mathbb{G}))$. Then their series representations in (6.6) and (6.8) are used to prove closure properties of the sequence $\{P_m\}_{m\geq 0}$ in $\mathcal{A}$.

7. Riccati equations over q-Banach algebras. The main result of this section is stated in Theorem 7.6, where it is shown that the unique solution of an LQR problem over a q-Banach algebra belongs to the same q-Banach algebra. In order to prove this result, we need to show that the unique solution of the corresponding Riccati equation also belongs to the same q-Banach algebras; this is the result of Theorem 7.3. The backbone of our analysis is built upon the existence of Riesz-spectral projection of Hamiltonian operators corresponding to Riccati equations over q-Banach algebras, which is discussed in Lemma 7.2.

The following result is well known and classic [14]. Suppose that $A, B, Q, R \in \mathcal{B}(\ell^2(\mathbb{G}))$ and operators Q and R are positive and strictly positive on $\ell^2(\mathbb{G})$, respectively. If (A, B) is exponentially stabilizable and $(A, Q^{1/2})$ is exponentially detectable, then the Riccati equation

$$A^*X + XA - XBR^{-1}B^*X + Q = 0 \tag{7.1}$$

has a unique strictly positive solution $X \in \mathcal{B}(\ell^2(\mathbb{G}))$. Furthermore, the closed-loop matrix $A_X = A + BK$ is exponentially stable in $\mathcal{B}(\ell^2(\mathbb{G}))$, where K is the LQR state feedback matrix given by

$$K = -R^{-1}B^*X. \tag{7.2}$$

For the closed-loop matrix $A_X = A - BR^{-1}B^*X$, there exist positive constants E and α such that

$$\left\|e^{tA_X}\right\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; E\, e^{-\alpha t} \quad \text{for all } t \geq 0. \tag{7.3}$$

Let Ω be the rectangle region in the complex plane with vertices

$$-\frac{\alpha}{2} \pm 2i\, \|A_X\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\,, \quad -2\, \|A_X\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \pm 2i\, \|A_X\|_{\mathcal{B}(\ell^2(\mathbb{G}))}\,. \tag{7.4}$$

The boundary of the rectangle region Ω in the complex plane is denoted by Γ. The following two lemmas are useful in the proof of the main results in Theorems 7.3 and 7.5.

LEMMA 7.1. *Suppose that A_X is exponentially stable in $\mathcal{B}(\ell^2(\mathbb{G}))$ and* (7.3) *holds for some positive constants E and α. Then*

$$\left\|(zI - A_X)^{-1}\right\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{2E}{\alpha} \tag{7.5}$$

and

$$\left\|(zI + A_X)^{-1}\right\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{E}{\alpha} \quad \textit{for all } z \in \Gamma, \tag{7.6}$$

where Γ is the boundary of the rectangle region Ω with vertices (7.4).

Proof. Let us consider the rectangle region in the complex plane with vertices (7.4) and boundary Γ. For every $z \in \Gamma$ with $\mathrm{Re}\{z\} \neq -\frac{\alpha}{2}$, we have that

$$|z| \;\geq\; 2\|A_X\|_{\mathcal{B}(\ell^2(\mathbb{G}))}.$$

This implies that $zI - A_X$ is invertible and

$$\left\|(zI - A_X)^{-1}\right\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{E}{\alpha}. \tag{7.7}$$

For every $z \in \Gamma$ with $\mathrm{Re}\{z\} = -\frac{\alpha}{2}$, it follows that

$$\big\|(zI - A_X)^{-1}\big\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{2E}{\alpha}. \tag{7.8}$$

By combining inequalities (7.7) and (7.8), we get

$$\big\|(zI - A_X)^{-1}\big\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{2E}{\alpha} \quad \text{for all } z \in \Gamma. \tag{7.9}$$

By a similar argument, we can show that $zI + A_X^*$ is invertible and

$$\big\|(zI + A_X^*)^{-1}\big\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \;\leq\; \frac{E}{\alpha} \tag{7.10}$$

for all $z \in \Gamma$. □

Lemma 7.2. *For $0 < q \leq 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$, $A, B, Q, R \in \mathcal{A}$, and Q and R are positive and strictly positive on $\ell^2(\mathbb{G})$, respectively. Assume that X is the solution of (7.1) and the corresponding closed-loop matrix A_X is exponentially stable. Let us define the Hamiltonian operator of the Riccati equation (7.1) by*

$$\mathbf{H} = \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix} \tag{7.11}$$

and set

$$\mathbf{E} \;:=\; \frac{1}{2\pi i} \int_\Gamma \big(z\mathbf{I}_2 - \mathbf{H}\big)^{-1}\, dz. \tag{7.12}$$

Then

$$\mathbf{E} \in \mathcal{M}_2(\mathcal{A}), \tag{7.13}$$

where $\mathcal{M}_2(\mathcal{A})$ is the algebra of 2×2 matrices with entries in $\mathcal{A}$ and is defined in Appendix 10.7. Moreover, we have

$$\mathbf{E} \;=\; \begin{bmatrix} I - ZX & Z \\ X(I - ZX) & XZ \end{bmatrix} \tag{7.14}$$

for some operator $Z \in \mathcal{B}(\ell^2(\mathbb{G}))$.

Proof. We denote the block identity matrix by

$$\mathbf{I}_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Through direct computations one can show that

$$z\mathbf{I}_2 - \mathbf{H} \;=\; \mathbf{T} \begin{bmatrix} zI - A_X & BR^{-1}B^* \\ 0 & zI + A_X^* \end{bmatrix} \mathbf{T}^{-1}, \tag{7.15}$$

where $z \in \Gamma$ and

$$\mathbf{T} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}.$$

The above identity together with (7.5) and (7.6) implies that $z\mathbf{I}_2 - \mathbf{H}$ is invertible in $\mathcal{B}(\ell^2(\mathbb{G}))$ for all $z \in \Gamma$. Therefore it follows from Theorem 4.6 and Lemma 10.1 that

$$(z\mathbf{I}_2 - \mathbf{H})^{-1} \in \mathcal{M}_2(\mathcal{A}) \tag{7.16}$$

for all $z \in \Gamma$.

For every given $z_0 \in \Gamma$, we have

$$\begin{aligned}(z\mathbf{I}_2 - \mathbf{H})^{-1} &= (z_0\mathbf{I}_2 - \mathbf{H})^{-1}\left(\mathbf{I}_2 - (z_0 - z)(z_0\mathbf{I}_2 - \mathbf{H})^{-1}\right)^{-1} \\ &= (z_0\mathbf{I}_2 - \mathbf{H})^{-1}\sum_{n=0}^{\infty}(z_0 - z)^n\left((z_0\mathbf{I}_2 - \mathbf{H})^{-1}\right)^n,\end{aligned} \tag{7.17}$$

where the series converges in $\mathcal{M}_2(\mathcal{A})$ whenever

$$K_0^{\frac{1}{q}}\ |z - z_0|\ \left\|(z_0\mathbf{I}_2 - \mathbf{H})^{-1}\right\|_{\mathcal{M}_2(\mathcal{A})}\ <\ 1.$$

This is because, according to (10.12),

$$\left\|(z_0 - z)^n\left((z_0\mathbf{I}_2 - \mathbf{H})^{-1}\right)^n\right\|_{\mathcal{M}_2(\mathcal{A})}^q \ \le\ |z_0 - z|^{qn}\ K_0^{n-1}\ \left\|(z_0\mathbf{I}_2 - \mathbf{H})^{-1}\right\|_{\mathcal{M}_2(\mathcal{A})}^{nq}. \tag{7.18}$$

Since Γ is a compact set, there exist finitely many points z_i, $i = 1, \ldots, N$, along with the four vertices of the rectangle region such that the collection of balls $B(z_i, r_i)$ with centers z_i with radii

$$r_i = \frac{1}{2}\Big(K_0^{\frac{1}{q}}\ \left\|(z_i\mathbf{I}_2 - \mathbf{H})^{-1}\right\|_{\mathcal{M}_2(\mathcal{A})}\Big)^{-1},\quad i = 1, \ldots, N,$$

is a covering of the boundary Γ. Therefore, we can write

$$\begin{aligned}\mathbf{E} &= \sum_{i=1}^{N}\ \int_{\Gamma_i}\ (z\mathbf{I}_2 - \mathbf{H})^{-1}\ dz \\ &= \sum_{i=1}^{N}\ (z_i\mathbf{I}_2 - \mathbf{H})^{-1}\ \sum_{n=0}^{\infty}\Big(\int_{\Gamma_i}(z_i - z)^n dz\Big)\ \Big((z_i\mathbf{I}_2 - \mathbf{H})^{-1}\Big)^n,\end{aligned} \tag{7.19}$$

where $\Gamma_i \subset \Gamma \cap B(z_i, r_i)$, $1 \le i \le N$, form a disjoint covering of the boundary Γ. According to (7.12), (7.15), (7.18), and (7.19), we conclude that

$$\mathbf{E} \in \mathcal{M}_2(\mathcal{A}). \tag{7.20}$$

From identity (7.15), it follows that

$$\mathbf{E}\ =\ \tfrac{1}{2\pi i}\,\mathbf{T}\Big(\int_\Gamma\ \begin{bmatrix}(zI - A_X)^{-1} & -(zI - A_X)^{-1}BR^{-1}B^*(zI + A_X^*)^{-1} \\ 0 & (zI + A_X^*)^{-1}\end{bmatrix} dz\Big)\,\mathbf{T}^{-1} \tag{7.21}$$

belongs to $\mathcal{B}(\ell^2(\mathbb{G}))$ by (7.5) and (7.6). Recall that Γ is the boundary of a rectangle region Ω such that the spectrum of A_X is contained in Ω and the closure of Ω is contained in the open left-half plane. Applying functional calculus to (7.21) leads to

$$\mathbf{E}\ =\ \mathbf{T}\begin{bmatrix} I & Z \\ 0 & 0\end{bmatrix}\mathbf{T}^{-1} = \begin{bmatrix} I - ZX & Z \\ X(I - ZX) & XZ\end{bmatrix}$$

for some operator $Z \in \mathcal{B}(\ell^2(\mathbb{G}))$. □

From the result of Lemma 7.2, we can immediately conclude that

$$I - ZX, Z, X(I - ZX), \quad XZ \in \mathcal{A}. \tag{7.22}$$

Our goal is to prove that $X \in \mathcal{A}$. In the following two theorems, we apply Lemma 7.2 along with several technical assumptions and develop a constructive proof based on finite covering of compact sets to show that the unique solution of an algebraic Riccati equation over a q-Banach algebra belongs to that q-Banach algebra.

THEOREM 7.3. *For $0 < q \le 1$, suppose that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$, $A, B, Q, R \in \mathcal{A}$, and Q and R are positive and strictly positive on $\ell^2(\mathbb{G})$, respectively. If we assume that* (i) *(A, B) is exponentially stabilizable and $(A, Q^{1/2})$ is exponentially detectable and that* (ii) *the dual Riccati equation*

$$AY + YA^* - YQY + BR^{-1}B^* = 0 \tag{7.23}$$

has a self-adjoint solution $Y \in \mathcal{B}(\ell^2(\mathbb{G}))$ such that $I + YX$ is invertible in $\mathcal{B}(\ell^2(\mathbb{G}))$, then the unique positive definite solution of the Riccati equation satisfies $X \in \mathcal{A}$ and A_X is exponentially stable on $\mathcal{A}$.

Proof. We have $\mathbf{HE} = \mathbf{EH}$, where $\mathbf{H}$ and $\mathbf{E}$ are defined in the statement of Lemma 7.2. It is straightforward to verify that Z in (7.22) satisfies the Lyapunov equation

$$A_X Z + Z A_X^* + BR^{-1}B^* = 0. \tag{7.24}$$

From our assumption (ii) and exponential stability property of A_X in $\mathcal{B}(\ell^2)$, the unique solution of the Lyapunov equation (7.24) can be represented as (see [12, Lemma 4.9] for more details)

$$Z = Y(I + XY)^{-1}, \tag{7.25}$$

where $Y = Y^* \in \mathcal{B}(\ell^2)$ is a solution of the dual Riccati equation (7.23). From (7.25), we have

$$I - ZX = I - Y(I + XY)^{-1}X = I - YX(I + YX)^{-1} = (I + YX)^{-1}. \tag{7.26}$$

According to our assumption (ii) and (7.26), it follows that $I - ZX$ is invertible in $\mathcal{B}(\ell^2(\mathbb{G}))$. From (7.22) and the inverse-closedness property of the q-Banach algebra $\mathcal{A}$ given in Theorem 4.6, we can conclude that $(I - ZX)^{-1} \in \mathcal{A}$, which together with $X(I - ZX) \in \mathcal{A}$ in (7.22) proves $X \in \mathcal{A}$. From the closure properties of $\mathcal{A}$, it also follows that

$$A_X = A - BR^{-1}B^*X \in \mathcal{A}.$$

Since A_X is exponentially stable on $\mathcal{B}(\ell^2(\mathbb{G}))$, we can conclude from Theorem 5.1 that it is also exponentially stable on $\mathcal{A}$. ∎

One can relax the second condition in Theorem 7.3 by assuming that the linear system is approximately controllable. The following definition is from [14, Definition 4.1.17].

DEFINITION 7.4. *The linear system* (3.1) *is approximately controllable if it is possible to steer from the origin to within an ϵ-neighborhood from every point in $\ell^2(\mathbb{G})$ for every arbitrary $\epsilon > 0$, i.e., if $\mathfrak{R}$ the reachable subspace of (A, B) is dense in $\ell^2(\mathbb{G})$, where*

$$\mathfrak{R} := \Big\{ z \in \ell^2(\mathbb{G}) \ \Big| \ \exists\ T > 0,\ u \in L^2\big([0,T]; \ell^2(\mathbb{G})\big) \text{ such that } z = \mathfrak{B}^T u \Big\},$$

and the controllability operator of (A, B) *is defined by*

$$\mathfrak{B}^T u := \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau \quad \textit{in} \quad \mathcal{B}(\ell^2(\mathbb{G})).$$

THEOREM 7.5. *For* $0 < q \leq 1$*, suppose that* $\mathcal{A}$ *is a proper* q*-Banach subalgebra of* $\mathcal{B}(\ell^2(\mathbb{G}))$*,* $A, B, Q, R \in \mathcal{A}$*, and* Q *and* R *are positive and strictly positive on* $\ell^2(\mathbb{G})$*, respectively. If we assume that* (A, B) *is approximately controllable and* $(A, Q^{1/2})$ *is exponentially detectable, then the unique positive definite solution of the Riccati equation satisfies* $X \in \mathcal{A}$ *and* A_X *is exponentially stable on* $\mathcal{A}$*.*

Proof. According to [14, Definition 4.1.17] and Definition 7.4, we can also rewrite the reachable subspace of (A, B) as

$$\mathfrak{R} = \bigcup_{T>0} \mathcal{R}(\mathfrak{B}^T), \tag{7.27}$$

where $\mathcal{R}(.)$ is the range of an operator. From [14, Lemma 4.1.6], it follows that

$$\mathcal{R}(\mathfrak{B}^T) = \mathcal{R}(\mathfrak{B}_K^T) \quad \text{for all } T > 0, \tag{7.28}$$

where $\mathfrak{B}_K^T$ is the controllability operator of $(A + BK, B)$. From (7.27) and (7.28), we can conclude that the reachable subspace of (A, B) is equal to $(A + BK, B)$ for any bounded feedback K. Therefore, (A, B) is approximately controllable if and only if $(A + BK, B)$ is for any bounded feedback K. This implies that (A_X, B) is also approximately controllable. Recall that $\mathbf{HE} = \mathbf{EH}$, where $\mathbf{H}$ and $\mathbf{E}$ are defined in the statement of Lemma 7.2. The matrix Z in (7.22) satisfies the Lyapunov equation (7.24). As A_X is exponentially stable in $\mathcal{B}(\ell^2)$, it follows that

$$\begin{aligned} Z &= -\int_0^\infty \frac{d}{dt}\left(e^{tA_X} Z e^{A_X^*}\right) dt \\ &= -\int_0^\infty e^{tA_X} \left(A_X Z + Z A_X^*\right) e^{tA_X^*}\, dt \\ &= \int_0^\infty e^{tA_X}\, BR^{-1}B^*\, e^{tA_X^*}\, dt \quad \text{in} \quad \mathcal{B}(\ell^2(\mathbb{G})). \end{aligned}$$

Since (A_X, B) is approximately controllable, according to [14, Theorem 4.1.22] we can conclude that Z is the unique strictly positive solution of (7.24).

In the next step, we prove that Z belongs to the q-Banach algebra $\mathcal{A}$. According to Lemma 7.2, we have $\mathbf{E} \in \mathcal{M}_2(\mathcal{A})$. From this result and (7.14), it immediately follows that

$$Z,\ XZ \in \mathcal{A}. \tag{7.29}$$

Since Z is strictly positive, we get that $Z^{-1} \in \mathcal{A}$ by the inverse closedness of the q-Banach algebra $\mathcal{A}$. This together with (7.29) proves that the solution X of the algebraic Riccati equation (7.1) belongs to $\mathcal{A}$. Based on a similar argument in the proof of Theorem 7.3, we can conclude that A_X is exponentially stable on $\mathcal{A}$. ∎

We remark that the result of Theorem 7.5 does not require the existence of a solution of the corresponding filter Riccati equation.

THEOREM 7.6. *Suppose that the assumptions of Theorem* 7.3 *or Theorem* 7.5 *hold. Then the LQR state feedback gain* (7.2) *belongs to* $\mathcal{A}$*.*

Proof. According to Theorem 7.3 or Theorem 7.5, $X \in \mathcal{A}$. Since $\mathcal{A}$ is closed under matrix multiplication and taking the inverse, we get that $K = -R^{-1}B^*X \in \mathcal{A}$. □

Remark 7.7. The proof of Theorem 7.3 is constructive, which enables us to obtain an upper bound for the q-norm of X over each covering set. By combining these upper bounds, one can only calculate a conservative upper bound for the q-norm of X. For this reason, we do not present this conservative upper bound.

Remark 7.8. Our proposed methodology in this paper can be applied to analyze other optimal control design problems with quadratic performance criteria, such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems [14]. The solutions of these optimal control problems involve solving two operator Riccati equations.

8. Applications to spatially distributed systems. We are interested in a class of linear systems for which there is a notion of spatial distance with respect to which coupling between every two subsystems decays in the spatial domain.

8.1. Admissible coupling weight functions. Our focus is on the class of (in)finite-dimensional linear systems (3.1)–(3.2) whose coupling structures are spatially decaying; i.e., the coupling strength between two subsystems decays by their spatial distance. The strength of coupling between two subsystems is modeled using a class of coupling weights.

DEFINITION 8.1. *A coupling weight w is a positive function on $\mathbb{G}\times\mathbb{G}$ that satisfies the following:* (i) $w(i,j) \geq 1$ *for all* $i,j \in \mathbb{G}$*;* (ii) $w(i,j) = w(j,i)$ *for all* $i,j \in \mathbb{G}$*; and* (iii) $\sup_{i\in\mathbb{G}} w(i,i) < \infty$.

The class of *submultiplicative coupling weights* is of particular interest,

$$w(i,j) \leq w(i,k)\, w(k,j) \quad \text{for all } i,j,k \in \mathbb{G}, \tag{8.1}$$

which helps us to define matrix norms that have a submultiplicative property. In order to prove some algebraic properties for matrix algebras in section 8.2, we need to impose some technical conditions on coupling weights to ensure certain growth rates.

DEFINITION 8.2. *Let $\rho : \mathbb{G} \times \mathbb{G} \longmapsto \mathbb{R}$ be a quasi distance. A coupling weight $w = [w(i,j)]_{i,j\in\mathbb{G}}$ is called admissible whenever there exist a companion weight $u = [u(i,j)]_{i,j\in\mathbb{G}}$, an exponent $\theta \in (0,1)$, and a positive constant D such that*

$$w(i,j) \leq w(i,k)\, u(k,j) + u(i,k)\, w(k,j) \quad \textit{for all } i,j,k \in \mathbb{G}, \tag{8.2}$$

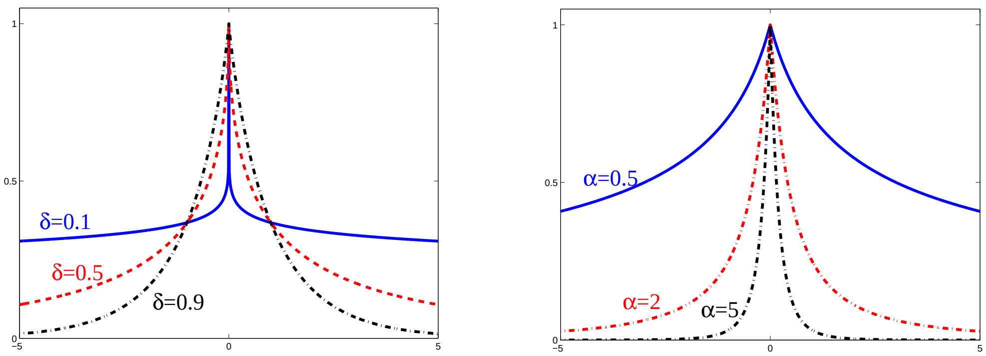
and the inequality

$$\sup_{i\in\mathbb{G}} \inf_{\tau\geq 0} \left\{ \Bigg[\sum_{\substack{j\in\mathbb{G}\\ \rho(i,j)<\tau}} |u(i,j)|^{\frac{2q}{2-q}} \Bigg]^{1-\frac{q}{2}} + t \sup_{\substack{j\in\mathbb{G}\\ \rho(i,j)\geq\tau}} \left(\frac{u(i,j)}{w(i,j)}\right)^q \right\} \leq Dt^{1-\theta} \tag{8.3}$$

holds for every $t \geq 1$ when $0 < q \leq 1$, and

$$\sup_{i\in\mathbb{G}} \inf_{\tau\geq 0} \left\{ \Bigg[\sum_{\substack{j\in\mathbb{G}\\ \rho(i,j)<\tau}} |u(i,j)|^2 \Bigg]^{\frac{1}{2}} + t \Bigg[\sum_{\substack{j\in\mathbb{G}\\ \rho(i,j)\geq\tau}} \left(\frac{u(i,j)}{w(i,j)}\right)^{\frac{q}{q-1}} \Bigg]^{\frac{q-1}{q}} \right\} \leq Dt^{1-\theta} \tag{8.4}$$

holds for every $t \geq 1$ when $1 < q \leq \infty$.



(a) The inverse of the subexponential coupling weights (8.6) with $d = 1$ and $\sigma = 1$ is depicted for three different values of parameter δ.

(b) The inverse of the polynomial coupling weights (8.7) with $d = 1$ and $\sigma = 1$ is depicted for three different values of parameter α.

FIG. 1. *The decay rates of coupling weights.*

We refer the reader to [42, page 3102] for a similar definition on the admissibility of a weight for $1 \le q \le \infty$. From Definition 8.2, a weak version of the submultiplicative property (8.1) can be deduced,

$$w(i,j) \le C_0 w(i,k) w(k,j) \quad \text{for all } i,j,k \in \mathbb{G}, \tag{8.5}$$

with constant $C_0 \le 2\sqrt[q]{D}$; see Appendix 10.2 for the proof. We should also emphasize that the class of admissible weights considered in this section is more general than the class of weights introduced earlier in [20, 41, 42].

In the next proposition, two popular coupling weights are shown to be admissible.

PROPOSITION 8.3. *Let us take the quasi distance $\rho(i,j) = \|i - j\|_\infty$ on $\mathbb{Z}^d$. For $0 < q \le 1$, the class of subexponential coupling weights*

$$e_{\sigma,\delta} := \left[e_{\sigma,\delta}(i,j)\right]_{i,j\in\mathbb{Z}^d} = \left[e^{\left(\frac{\|i-j\|_\infty}{\sigma}\right)^\delta}\right]_{i,j\in\mathbb{Z}^d} \tag{8.6}$$

with parameters $\sigma > 0$ and $\delta \in (0,1)$ is submultiplicative and admissible with constants

$$D_e = 2^{d+1-\frac{dq}{2}}, \quad \theta_e = \frac{q\delta(2-2^\delta)}{q\delta + d(1-\frac{q}{2})\sigma^\delta},$$

and the class of polynomial coupling weights

$$\pi_{\alpha,\sigma} \ := \ \left[\pi_{\alpha,\sigma}(i,j)\right]_{i,j\in\mathbb{Z}^d} \ = \ \left[\left(\frac{1+\|i-j\|_\infty}{\sigma}\right)^\alpha\right]_{i,j\in\mathbb{Z}^d} \tag{8.7}$$

with parameters $\alpha, \sigma > 0$ is submultiplicative and admissible with constants

$$D_\pi = 2^{q\alpha+1} \max\left\{1, (2\sigma)^{d(1-\frac{q}{2})}\right\}, \quad \theta_\pi = \frac{q\alpha}{q\alpha + d(1-\frac{q}{2})}.$$

Subexponential weights and polynomial weights appear in the modeling of several real-world systems (cf. Example 2.2 in [42]). For a specific system in real-word applications, the decay rate of the elements of its state-space matrices determines the type of coupling weight that is suitable. Figures 1a and 1b depict various decay rates for the two coupling functions discussed in Proposition 8.3.

8.2. Gröchenig–Schur matrix algebras. One of the interesting and practical examples of a q-Banach algebra is $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \le 1$.

DEFINITION 8.4. *For a given admissible coupling weight w on $\mathbb{G} \times \mathbb{G}$, the Gröchenig–Schur class of matrices on $\mathbb{G}$, to be denoted by $\mathcal{S}_{q,w}(\mathbb{G})$, is defined as*

$$\mathcal{S}_{q,w}(\mathbb{G}) = \left\{ A = [a_{ij}]_{i,j \in \mathbb{G}} \;\middle|\; \|A\|_{\mathcal{S}_{q,w}(\mathbb{G})} < \infty \right\} \tag{8.8}$$

for $0 < q \le \infty$, where the $\mathcal{S}_{q,w}$-measure for $0 < q < \infty$ is defined by

$$\|A\|_{\mathcal{S}_{q,w}(\mathbb{G})} := \max \left\{ \sup_{i \in \mathbb{G}} \left(\sum_{j \in \mathbb{G}} |a_{ij}|^q w(i,j)^q \right)^{1/q}, \; \sup_{j \in \mathbb{G}} \left(\sum_{i \in \mathbb{G}} |a_{ij}|^q w(i,j)^q \right)^{1/q} \right\} \tag{8.9}$$

and for $q = \infty$ by

$$\|A\|_{\mathcal{S}_{\infty,w}(\mathbb{G})} := \sup_{i,j \in \mathbb{G}} |a_{ij}| w(i,j). \tag{8.10}$$

It should be remarked that this class of matrices was studied for $q = 1$ in [20, 31, 32], for $q = \infty$ in [25], and for $1 \le q \le \infty$ in [41, 42]. In this paper, their application is extended to include the range of exponents $0 < q \le 1$. In the above extension, the quantity (8.9) is a q-norm according to Definition 4.1 and it is nonconvex, as the ℓ^q-measures are nonconvex for this range of exponents. Therefore, the space of matrices $\mathcal{S}_{q,w}(\mathbb{G})$ is not locally convex and is not a Banach space [33]. On some occasions when it is not ambiguous, we will employ $\mathcal{S}_{q,w}$ instead of $\mathcal{S}_{q,w}(\mathbb{G})$ for simplicity of notation.

THEOREM 8.5. *For every $0 < q \le 1$ and admissible coupling weight function w, the space of matrices $\mathcal{S}_{q,w}(\mathbb{G})$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$.*

This result shows that although the space of matrices $\mathcal{S}_{q,w}(\mathbb{G})$ is not a Banach space for $0 < q < 1$, it still exhibits some useful algebraic properties that will enable us to study algebraic properties of linear systems over such nonconvex matrix spaces.

As we will discuss later on in section 8.5, the space of sparse matrices $\mathcal{S}_{0,1}(\mathbb{G})$ is not inverse-closed in general and it can be reasonably approximated by $\mathcal{S}_{q,w}(\mathbb{G})$ for small enough q. A relevant question here is whether $\mathcal{S}_{q,w}(\mathbb{G})$ is inverse-closed for $0 < q < 1$. It is known that $\mathcal{S}_{q,w}(\mathbb{G})$ for $1 \le q \le \infty$ is an inverse-closed Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$ under certain assumptions on the weight function w. There are several families of Banach subalgebras of (infinite-dimensional) matrices with some certain off-diagonal decay properties for which the inverse-closedness property, also known as Wiener's lemma, is inherited from the parent Banach algebra. We refer the reader to [2, 6, 19, 20, 21, 36, 40, 41, 42, 44] and the survey papers [18, 28, 37] for more details. The conclusions of Theorems 8.5 and 4.6 imply that $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \le 1$ is indeed inverse-closed in $\mathcal{B}(\ell^2(\mathbb{G}))$.

8.3. Spatially decaying linear systems. We are interested in a class of linear systems for which there is a notion of spatial distance with respect to which coupling between every two subsystems decays in the spatial domain.

DEFINITION 8.6. *For a fixed exponent $0 < q \le \infty$, a linear system* (3.1)–(3.2) *is called spatially decaying on $\mathbb{G}$ with respect to an admissible coupling weight w if $A, B, C, D \in \mathcal{S}_{q,w}(\mathbb{G})$.*

These are systems with off-diagonally decaying state-space matrices. This definition generalizes the earlier works [31, 32] that only studied the class of spatially decaying systems for $q = 1$. For every exponent $0 < q < 1$, the corresponding class of spatially decaying systems is defined over a matrix space $\mathcal{S}_{q,w}(\mathbb{G})$ that is not a Banach space, which prevents us from applying existing techniques in the literature that are based on continuous integrals on Banach spaces. In the following sections, we develop new tools to study algebraic properties of linear systems and their corresponding optimal control problems over $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q < 1$.

Example 8.7. We consider the heat equation on an infinite bar with spatially varying thermal diffusivity coefficient $\lambda(s)$ and distributed heat injection $u(s,t)$ as its control input:

$$\frac{\partial}{\partial t}\psi(s,t) = \lambda(s)\frac{\partial^2}{\partial s^2}\psi(s,t) + u(s,t), \tag{8.11}$$

$$y(s,t) = \psi(s,t) \tag{8.12}$$

for all $s \in \mathbb{R}$ and $t \geq 0$, where the underlying spatial domain is $\mathbb{R}$. The state variable $\psi(s,t)$ describes the temperature of the bar at a given spatial location s at time t, and the initial temperature distribution is given by $\psi(s,0) = f(s)$. Suppose that the linear system (8.11)–(8.12) has a unique solution under suitable assumptions [23]. This system can be spatially discretized using the following centered difference approximation at each instant of time t and spatial locations s:

$$\frac{\partial^2}{\partial s^2}\psi(s,t) = \frac{\psi(s+h,t) - 2\psi(s,t) + \psi(s-h,t)}{h^2} \; + \; O(h^2) \tag{8.13}$$

for sufficiently small $h > 0$. For a suitable step size $h > 0$, let us denote the state of the discretized system by $x(t) = [\ \ldots, x_{k-1}(t), x_k(t), x_{k+1}(t), \ldots\ ]^*$, where $x_k(t) = \psi(kh,t)$ for all $k \in \mathbb{Z}$. By defining localized control inputs $u_k(t) = u(kh,t)$ and output measurements $y_k(t) = y(kh,t)$, the resulting localized discretized linear systems can be written as

$$\dot{x}_k(t) = h^{-2}\lambda_k\big(x_{k-1}(t) - 2x_k(t) + x_{k+1}(t)\big) + u_k(t), \tag{8.14}$$

$$y_k(t) = x_k(t), \tag{8.15}$$

where $\lambda_k = \lambda(kh)$. The overall dynamics of the spatially discretized model can be cast in the form of (3.1)–(3.2) with state-space matrices $B = C = I$, $D = 0$, and

$$A = h^{-2}\begin{bmatrix} & & \ddots & \ddots & & \vdots & \vdots & & \\ \ldots & 0 & \lambda_{k-1} & -2\lambda_{k-1} & \lambda_{k-1} & 0 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \lambda_k & -2\lambda_k & \lambda_k & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & \lambda_{k+1} & -2\lambda_{k+1} & \lambda_{k+1} & 0 & \ldots \\ & \vdots & \vdots & & & \ddots & \ddots & & \end{bmatrix} \tag{8.16}$$

and initial condition $x(0) = [\ \ldots, f((k-1)h), f(kh), f((k+1)h), \ldots\ ]^*$. Since all state-space matrices are banded, $A, B, C, D \in \mathcal{S}_{q,w}(\mathbb{G})$ for all admissible coupling weights w and exponents $0 < q \leq \infty$. Therefore, the discretized model is spatially decaying according to Definition 8.6.

Example 8.8. Let us consider the following infinite-dimensional platoon model on spatial domain $\mathbb{Z}$:

$$\dot{x}_i(t) = \sum_{j\in\mathbb{Z}} d_{ij}\,(x_j(t) - x_i(t)) \; + \; b_i u_i(t), \tag{8.17}$$

$$y_i(t) = c_i x_i(t), \tag{8.18}$$

where $d_{ij} = \delta_{ij}(1 + |i - j|)^{-\alpha}$ for some $\alpha > 1/q$. For every $i, j \in \mathbb{Z}$, the coefficient δ_{ij} is drawn from a standard normal distribution. This simple spatially varying platoon model is inspired by the flocking and platoon models introduced in [9, 26]. The class of spatially invariant platoons is considered in [11]. The dynamics of the overall platoon can be rewritten in the form of (3.1)–(3.2) with the infinite-dimensional state-space matrices $A = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = \begin{cases} d_{ij} & \text{if } i \neq j, \\ d_{ii} - \sum_{j\in\mathbb{Z}} d_{ij} & \text{if } i = j, \end{cases} \tag{8.19}$$

$B = \operatorname{diag}(\ldots, b_{i-1}, b_i, b_{i+1}, \ldots)$, $C = \operatorname{diag}(\ldots, c_{i-1}, c_i, c_{i+1}, \ldots)$, and $D = 0$. Suppose that $\pi_{\alpha',1}$ is given by (8.7) with $0 < \alpha' < \alpha - 1/q$. It is straightforward to verify that $A, B, C, D \in \mathcal{S}_{q,\pi_{\alpha',1}}(\mathbb{G})$.

8.4. Spatial decay properties of the LQR feedback. In [32], the authors proposed an operator-theoretic approach based on Banach algebras of spatially decaying matrices $\mathcal{S}_{1,w}(\mathbb{G})$ to analyze the spatial structure of the optimal solution of the LQR problem for spatially distributed systems over $\mathcal{S}_{1,w}(\mathbb{G})$ and aimed at showing that LQR controllers inherit spatial decay properties of the underlying systems. The results of [10, 32] state that the LQR state feedback gain $K = \left[k_{ij}\right]_{i,j\in\mathbb{G}}$ is spatially localized, i.e.,

$$|k_{ij}| \leq C_0 w(i,j)^{-1}, \quad i, j \in \mathbb{G}, \tag{8.20}$$

where $C_0 = \|K\|_{\mathcal{S}_{\infty,w}(\mathbb{G})} \leq \|K\|_{\mathcal{S}_{1,w}(\mathbb{G})}$ is a finite number. This implies that the underlying information structure of the LQR controller is sparse and spatially localized on the spatial domain and each local controller needs to receive state information only from some neighboring subsystems, rather than from the entire network.

If we assume that linear system (3.1)–(3.2) is spatially decaying according to Definition 8.6, then the conclusions of Theorems 6.1 and 7.3 also hold for $\mathcal{S}_{q,w}(\mathbb{G})$ for all $0 < q \leq 1$. For $1 \leq q \leq \infty$, $\mathcal{S}_{q,w}(\mathbb{G})$ is a Banach algebra. Our proof does not depend on whether the underlying space is a Banach space or not. Therefore, our proofs are still valid for all exponents $0 < q \leq \infty$. Moreover, the results of Theorems 8.5 and 7.6 imply that the LQR state feedback gain satisfies

$$K \in \mathcal{S}_{q,w}(\mathbb{G}) \quad \text{for all } 0 < q \leq 1.$$

This result states that the LQR state feedback gain $K = [k_{ij}]_{i,j\in\mathbb{G}}$ is spatially localized, i.e.,

$$|k_{ij}| \ \leq \ C_0\, w(i,j)^{-1}, \quad i, j \in \mathbb{G}, \tag{8.21}$$

where $C_0 = \|K\|_{\mathcal{S}_{\infty,w}(\mathbb{G})} \leq \|K\|_{\mathcal{S}_{q,w}(\mathbb{G})}$ is a finite number. This result motivates us to utilize the $\mathcal{S}_{q,w}$-measure as a viable tool to study sparsity and spatial localization features of spatially decaying systems.

8.5. The space of sparse matrices and their asymptotic approximations. The range of exponents $0 < q < 1$ is essential for our development, as we can asymptotically quantify the sparsity and spatial localization properties of spatially decaying systems for sufficiently small values of exponent q. This comes at the expense of working with nonconvex matrix spaces. Let us consider (in)finite-dimensional vectors $x = [x_i]_{i\in\mathbb{G}}$ with bounded entries. The ℓ^q-measure approximates the ℓ^0-measure defined by (2.1) asymptotically, i.e.,

$$\lim_{q\to 0} \ \|x\|_{\ell^q(\mathbb{G})}^q \ = \ \|x\|_{\ell^0(\mathbb{G})}. \tag{8.22}$$

This observation motivates us to consider the asymptotic behavior of the $\mathcal{S}_{q,w}$-measure as q tends to zero. For a given matrix $A = [a_{ij}]_{i,j\in\mathbb{G}}$, we define an ideal sparsity measure, the so-called $\mathcal{S}_{0,1}$-measure, by

$$\|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} := \max\left\{\sup_{i\in\mathbb{G}} \|a_{i\cdot}\|_{\ell^0(\mathbb{G})}, \sup_{j\in\mathbb{G}} \|a_{\cdot j}\|_{\ell^0(\mathbb{G})}\right\}, \tag{8.23}$$

where $a_{i\cdot}$ is the ith row and $a_{\cdot j}$ is the jth column of matrix A. The value of the $\mathcal{S}_{0,1}$-measure gives the maximum number of nonzero entries in all rows and columns of matrix A.

THEOREM 8.9. *For a given matrix $A = [a_{ij}]_{i,j\in\mathbb{G}}$ with a finite $\mathcal{S}_{0,1}$-measure and bounded entries, i.e., $\|A\|_{\mathcal{S}_{0,1}} < \infty$ and $\|A\|_{\mathcal{S}_{\infty,w}} < \infty$, we have*

$$\lim_{q\to 0} \|A\|^q_{\mathcal{S}_{q,w}(\mathbb{G})} = \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}. \tag{8.24}$$

A notable implication of this theorem is that small values of q (closer to zero) can lead to reasonable approximations of the ideal sparsity measure for matrices. This is particularly true for spatially decaying matrices with slowly decaying rates, such as polynomially decaying matrices. However, for matrices with rapidly decaying rates, such as subexponentially decaying matrices, larger values of q (closer to one) can also result in reasonable measures for sparsity; see section 8.7 for more details.

The proposed $\mathcal{S}_{0,1}$-measure has an interesting interpretation when A is the adjacency matrix of an unweighted undirected graph. In this case, the value of the $\mathcal{S}_{0,1}$-measure is equal to the maximum node degree of the graph. The sparsity $\mathcal{S}_{0,1}$-measure has several advantages over the conventional sparsity measure:

$$\|A\|_0 = \sum_{i,j\in\mathbb{G}} |a_{ij}|^0 = \operatorname{card}\left\{a_{ij} \neq 0 \;\middle|\; i,j \in \mathbb{G}\right\}. \tag{8.25}$$

For sparse matrices, the value of the $\mathcal{S}_{0,1}$-measure reveals some valuable information about sparsity as well as spatial locality features of a matrix, while (8.25) only gives away the total number of nonzero elements of a matrix. Moreover, (8.25) does not exhibit any interesting algebraic property and is not useful in infinite-dimensional settings. Let us consider the set of all sparse matrices characterized by

$$\mathcal{S}_{0,1}(\mathbb{G}) = \left\{A \;\middle|\; \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} < \infty\right\}. \tag{8.26}$$

PROPOSITION 8.10. *For all nonzero scalars α and matrices $A, B \in \mathcal{S}_{0,1}(\mathbb{G})$, the sparsity measure* (8.23) *satisfies the following properties:*

(i) $\|\alpha A\|_{\mathcal{S}_{0,1}(\mathbb{G})} = \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}$;
(ii) $\|A + B\|_{\mathcal{S}_{0,1}(\mathbb{G})} \leq \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} + \|B\|_{\mathcal{S}_{0,1}(\mathbb{G})}$; *and*
(iii) $\|AB\|_{\mathcal{S}_{0,1}(\mathbb{G})} \leq \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}\|B\|_{\mathcal{S}_{0,1}(\mathbb{G})}$.

Properties (i)–(iii) imply that the set $\mathcal{S}_{0,1}(\mathbb{G})$ is closed under addition and scalar multiplication. One of the fundamental properties of $\mathcal{S}_{q,w}(\mathbb{G})$ that is not inherited by $\mathcal{S}_{0,1}(\mathbb{G})$ is the inverse-closedness property; a formal definition of inverse-closedness is given in Definition 4.5. For instance, a Toeplitz band matrix belongs to $\mathcal{S}_{0,1}(\mathbb{G})$, while its inverse may not live in $\mathcal{S}_{0,1}(\mathbb{G})$. In fact, $\mathcal{S}_{0,1}(\mathbb{G})$ is not even an algebra. In sections 5, 6, and 7, we show that the inverse-closedness property of the algebras of spatially decaying matrices plays a central role in exploiting various structural properties of spatially decaying systems.

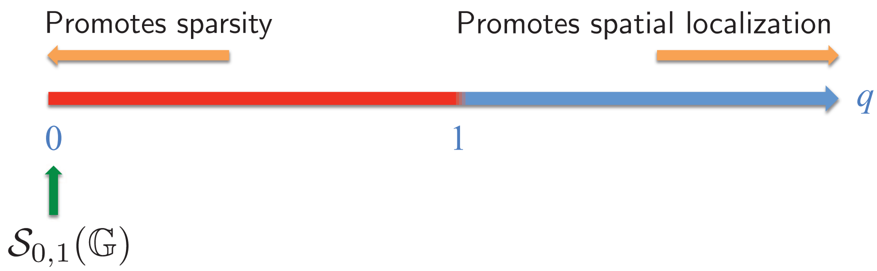


FIG. 2. *For $1 \le q \le \infty$, $\mathcal{S}_{q,w}(\mathbb{G})$ is a Banach algebra. However, $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q < 1$ is not a Banach space, where in this case analysis of spatially decaying systems over $\mathcal{S}_{q,w}(\mathbb{G})$ requires new techniques based on notions of q-Banach algebras.*

Remark 8.11. For large values of q (closer to ∞), the $\mathcal{S}_{q,w}$-measure asymptotically approximates the $\mathcal{S}_{\infty,w}$-measure and its value measures the degree of spatial localization; see Figure 2. On the other end of the spectrum, the $\mathcal{S}_{q,w}$-measure asymptotically approximates the $\mathcal{S}_{0,1}$-measure for small values of q (closer to 0). This observation implies that in order to study sparsity features of spatially decaying systems it is necessary to consider the range of exponents $0 < q \le 1$. For mathematical purposes, $q = 1$ is the best possible choice, as the $\mathcal{S}_{1,w}$-measure is the closest convex estimate of the $\mathcal{S}_{0,1}$-measure. Furthermore, spatially decaying matrices in $\mathcal{S}_{1,w}(\mathbb{G})$ enjoy the fastest decay rates among all families of $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q \le 1$.

8.6. Fundamental tradeoffs between stability margin and performance loss. In the rest of this section, in order to present our results in more explicit forms, we will limit our analysis to matrices that are defined on $\mathbb{G} = \mathbb{Z}$ endowed with quasi-distance function $\rho(i,j) = |i-j|$. In order to characterize a fundamental limit on interplay between stability margins and optimal performance loss in spatially distributed systems, we approximate the corresponding LQR feedback gain K by a sparse feedback gain $K^{\mathfrak{T}}$ for a given truncation length $\mathfrak{T} > 0$ as follows:

$$\left(K^{\mathfrak{T}}\right)_{ij} = \begin{cases} K_{ij} & \text{if } |i-j| \le \mathfrak{T}, \\ 0 & \text{if } |i-j| > \mathfrak{T}. \end{cases} \tag{8.27}$$

THEOREM 8.12. *For $0 < q \le 1$, suppose that $K \in \mathcal{S}_{q,w}(\mathbb{Z})$ and $K^{\mathfrak{T}}$ is defined by* (8.27). *Then*

$$\|K - K^{\mathfrak{T}}\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \;\le\; C_0\, w(\mathfrak{T})^{-1}, \tag{8.28}$$

where $C_0 = \|K\|_{\mathcal{S}_{q,w}(\mathbb{Z})}$ and $w(\mathfrak{T}) = \inf_{|i-j|>\mathfrak{T}} w(i,j)$.

Proof. We apply the Schur test to estimate the operator norm in $\mathcal{B}(\ell^2(\mathbb{Z}))$, i.e.,

$$\|K - K^{\mathfrak{T}}\|^2_{\mathcal{B}(\ell^2(\mathbb{Z}))} \le \left(\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|K_{ij} - (K^{\mathfrak{T}})_{ij}\right|\right) \left(\sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \left|K_{ij} - (K^{\mathfrak{T}})_{ij}\right|\right).$$

First, let us consider the following term:

$$\begin{aligned} \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|K_{ij} - (K^{\mathfrak{T}})_{ij}\right| &= \sup_{i\in\mathbb{Z}} \sum_{|i-j|>\mathfrak{T}} \left|K_{ij}\right| = \sup_{i\in\mathbb{Z}} \sum_{|i-j|>\mathfrak{T}} \left|K_{ij}\right| w(i,j) w(i,j)^{-1} \\ &\le \left(\sup_{i\in\mathbb{Z}} \sum_{|i-j|>\mathfrak{T}} \left|K_{ij}\right| w(i,j)\right) w(\mathfrak{T})^{-1} \le C_0\, w(\mathfrak{T})^{-1}. \end{aligned} \tag{8.29}$$

The last inequality holds due to the monotonic inequality $\|z\|_1 \leq \|z\|_q$ for all $z \in \ell^q(\mathbb{Z})$ and $0 < q \leq 1$. Likewise, by interchanging indices i and j a similar bound can be obtained. Therefore, it completes the proof of the conclusion (8.28). □

For the class of subexponential coupling weights with parameters $\sigma > 0$ and $\delta \in (0,1)$, the inequality (8.28) becomes

$$\|K - K^{\mathfrak{T}}\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \;\leq\; C_{\mathrm{e}}\; e^{-(\mathfrak{T}/\sigma)^{\delta}},$$

where $C_{\mathrm{e}} = \|K\|_{\mathcal{S}_{q,e_{\sigma,\delta}}(\mathbb{Z})}$. Similarly, the error bound for the class polynomial coupling weight functions with parameters $\alpha, \sigma > 0$ is

$$\|K - K^{\mathfrak{T}}\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \;\leq\; C_{\pi}\,(\mathfrak{T}/\sigma)^{-\alpha},$$

where $C_{\pi} = \|K\|_{\mathcal{S}_{q,\pi_{\alpha,\sigma}}(\mathbb{Z})}$.

The inequality (8.28) in Theorem 8.12 implies that for spatially decaying state feedback gains the truncation tail $K - K^{\mathfrak{T}}$ can be made arbitrarily small as $\mathfrak{T}$ gets large. This property enables us to characterize stabilizing truncated state feedback gains by using the small-gain stability argument. Let us consider the truncated closed-loop system

$$\dot{x} = (A + BK^{\mathfrak{T}})x. \tag{8.30}$$

One can decompose this system as two subsystems:

$$\dot{x} = (A + BK)x + w, \tag{8.31}$$

$$w = B(K^{\mathfrak{T}} - K)x. \tag{8.32}$$

Since K is the LQR feedback gain, the LQR closed-loop operator $A + BK$ in (8.31) is exponentially stable. If $B \in \mathcal{B}(\ell^2(\mathbb{G}))$, we can apply the small-gain theorem and characterize a fundamental limit in the form of a lower bound for stabilizing truncation lengths for different admissible coupling weights. For the class of subexponential coupling weights with parameters $\sigma > 0$ and $\delta \in (0,1)$, the truncated feedback gain $K^{\mathfrak{T}}$ is exponentially stabilizing if the truncation length $\mathfrak{T}$ satisfies the following inequality:

$$\mathfrak{T} \;>\; \mathfrak{T}_s = \sigma\left(\log\left(C_{\mathrm{e}}\,\|B\|_{\mathcal{B}(\ell^2(\mathbb{Z}))}\sup_{\mathrm{Re}(s)>0}\sigma_{\max}(G(s))\right)\right)^{\frac{1}{\delta}} \tag{8.33}$$

and $G(s) = (sI - (A + BK))^{-1}$ is the transfer function of the closed-loop system. The stabilizing truncation length for the class of polynomial coupling weights with parameters $\alpha, \sigma > 0$ is given by

$$\mathfrak{T}_s \;=\; \sigma\left(C_{\pi}\,\|B\|_{\mathcal{B}(\ell^2(\mathbb{Z}))}\sup_{\mathrm{Re}(s)>0}\sigma_{\max}(G(s))\right)^{\frac{1}{\alpha}}.$$

A fundamental tradeoff emerges between truncation length and performance loss; i.e., larger truncation lengths result in smaller performance loss. Let us consider the following quadratic cost functional:

$$J(x_0, K) = \int_0^{\infty} (x^* Q x + x^* K^* R K x)\, dt, \tag{8.34}$$

where x_0 is the initial condition of the linear system and K an exponentially stabilizing state feedback gain. It is straightforward to show that

$$J(x_0, K) = x_0^* X x_0, \tag{8.35}$$

in which X is the unique strictly positive definite solution of the Lyapunov equation

$$(A + BK)^* X + X(A + BK) + Q + K^* RK = 0. \tag{8.36}$$

We define the performance loss measure by the following quantity:

$$\Pi_K(\mathfrak{T}, x_0) \ = \ J(x_0, K^{\mathfrak{T}}) - J(x_0, K). \tag{8.37}$$

One of the important remaining problems is to study the asymptotic behavior of the performance loss $\Pi_K(\mathfrak{T}, x_0)$ as $\mathfrak{T}$ tends to infinity. This problem is beyond the scope of this paper and will be addressed in our future works.

8.7. Near-ideal and optimal degrees of sparsity and spatial localization. In this section, we propose a method to compute a value for parameter $0 < q < 1$ such that the $\mathcal{S}_{q,w}$-measure approximates the $\mathcal{S}_{0,1}$-measure in probability. In order to present our results in more explicit and sensible forms, we will limit our analysis to the class of subexponentially decaying matrices that are defined on $\mathbb{G} = \mathbb{Z}$.

Definition 8.13. *For a given truncation threshold $\epsilon > 0$ and matrix K, the threshold matrix of K is denoted by K_ϵ and defined by setting $(K_\epsilon)_{ij} = 0$ if $|K_{ij}| < \epsilon$ and $(K_\epsilon)_{ij} = K_{ij}$ otherwise.*

In order to present our results in more explicit and sensible forms, only in this section, we limit our focus to the class of subexponentially decaying random matrices of the form

$$\mathcal{R}_{\sigma,\delta}(\mathbb{Z}) = \left\{ K = \left[r_{ij} \exp\left(- \left(\frac{|i-j|}{\sigma} \right)^{\delta} \right) \right]_{i,j \in \mathbb{Z}} \middle| \ r_{ij} \sim \mathbf{U}(-1, 1) \right\} \tag{8.38}$$

for some given parameters $\sigma > 0$ and $\delta \in (0, 1)$. The coefficients r_{ij} are drawn from the continuous uniform distribution $\mathbf{U}(-1, 1)$. It is assumed that the underlying spatial domain is $\mathbb{Z}$ and that the spatial distance between node i and j is measured by $|i - j|$. The corresponding admissible coupling weight function for this class of spatially decaying matrices is given by

$$e_{\sigma',\delta} := \left[e_{\sigma',\delta}(i, j) \right]_{i,j \in \mathbb{Z}} = \left[\exp\left((|i - j|/\sigma')^{\delta} \right) \right]_{i,j \in \mathbb{Z}} \tag{8.39}$$

for some $\sigma' > \sigma$.

Definition 8.14. *For a given truncation threshold $0 < \epsilon < 1$, the sparsity indicator function for the class of random subexponentially decaying matrices $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ is defined by*

$$\Psi_{K,w}(q, \epsilon) \ := \ \frac{\|K\|^q_{\mathcal{S}_{q,w}}}{2 \left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1}, \tag{8.40}$$

where $\lfloor . \rfloor$ is the floor function.

The sparsity indicator function provides a reasonable criterion for calculating proper values for exponent q and parameters in the weight in order to measure sparsity of matrices in $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ using the $\mathcal{S}_{q,w}$-measure. This is simply because of the following inequality that shows that the value of the $\mathcal{S}_{0,1}$-measure of the threshold matrix of a matrix $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ can be upper bounded by

$$\|K_\epsilon\|_{\mathcal{S}_{0,1}} \;\leq\; 2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1. \tag{8.41}$$

THEOREM 8.15. *Suppose that $w_0 \equiv 1$ is the trivial weight and parameters $\sigma > 0$ and $\delta \in (0,1)$ are given. Let us define parameter $\beta = q \ln \epsilon^{-1}$. For every $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$, the sparsity indicator function $\Psi_{K,w_0}(q,\epsilon)$ converges to $\gamma := \beta^{-s}\Gamma(s+1)$ in probability as the truncation threshold ϵ tends to zero, i.e.,*

$$\lim_{\epsilon \to 0^+} \mathbb{P}\Big\{ \left|\Psi_{K,w_0}(q,\epsilon) - \gamma\right| < \epsilon_0 \Big\} \;=\; 1 \tag{8.42}$$

for every $\epsilon_0 > 0$, where $s = \delta^{-1}$ and $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$ is the Gamma function.

Proof. The exponent $q > 0$ can be related to another auxiliary variable $\beta > 0$ using the equation $\epsilon^q = e^{-\beta}$, from which we get $\beta = q \ln \epsilon^{-1}$. For every $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$, one can verify that

$$\begin{aligned} \Psi_{K,w_0}(q,\epsilon) \;&\leq\; \frac{1}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1} \sum_{k \in \mathbb{Z}} e^{-q\left|\frac{k}{\sigma}\right|^\delta} \\ &=\; \frac{2\sigma \sqrt[\delta]{\ln \epsilon^{-1}}}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1} \, \frac{1}{2\sigma \sqrt[\delta]{\ln \epsilon^{-1}}} \sum_{k \in \mathbb{Z}} e^{-\beta\left|\frac{k}{\sigma \sqrt[\delta]{\ln \epsilon^{-1}}}\right|^\delta}. \end{aligned} \tag{8.43}$$

It is straightforward to show that

$$\lim_{\epsilon \to 0} \frac{2\sigma \sqrt[\delta]{\ln \epsilon^{-1}}}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1} \;=\; 1 \tag{8.44}$$

and

$$\frac{1}{2\sigma \sqrt[\delta]{\ln \epsilon^{-1}}} \sum_{k \in \mathbb{Z}} e^{-\beta\left|\frac{k}{\sigma \sqrt[\delta]{\ln \epsilon^{-1}}}\right|^\delta} \;\longrightarrow\; \frac{1}{2}\int_{-\infty}^{\infty} e^{-\beta|t|^\delta} dt = \frac{1}{\delta}\int_0^\infty e^{-\beta u} u^{\frac{1}{\delta}-1} du = \gamma \tag{8.45}$$

as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon \to 0} \Psi_{K,w_0}(q,\epsilon) \;\leq\; \int_0^\infty e^{-\beta t^\delta} dt = \gamma. \tag{8.46}$$

On the other hand, the expected value of the sparsity indicator function is lower bounded as follows:

$$\begin{aligned} \mathbb{E}\left[\Psi_{K,w_0}(q,\epsilon)\right] &\geq \frac{1}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1} \sum_{k \in \mathbb{Z}} \mathbb{E}\left[|r_{0j}|^q\right] e^{-q\frac{|0-k|^\delta}{\sigma^\delta}} \\ &=\; \frac{1}{1+q} \, \frac{1}{2\left\lfloor \sigma \sqrt[\delta]{\ln \epsilon^{-1}} \right\rfloor + 1} \sum_{k \in \mathbb{Z}} e^{-\beta\left|\frac{k}{\sigma \sqrt[\delta]{\ln \epsilon^{-1}}}\right|^\delta}. \end{aligned}$$

From (8.44) and (8.45), we have that

$$\lim_{\epsilon\to 0}\mathbb{E}\big[\Psi_{K,w_0}(q,\epsilon)\big] \;\geq\; \int_0^\infty e^{-\beta t^\delta}dt = \gamma. \tag{8.47}$$

In (8.47), if we keep the auxiliary parameter β fixed, then $q \to 0$ whenever $\epsilon \to 0$. The two inequalities in limits (8.46) and (8.47) prove (8.42), which shows the convergence of the sparsity indicator function $\Psi_{K,w_0}(q,\epsilon)$ in probability. □

The result of Theorem 8.15 asserts that the sparsity indicator functions associated with the class of subexponentially decaying random matrices $\mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ can be made arbitrarily close to a number close to one (i.e., $\gamma = 1$ or $\beta = \big(\Gamma(\delta^{-1}+1)\big)^{\delta}\big)$ in probability when we use trivial weight w_0. In the following result, it is shown that one can select the exponent q and the nontrivial weight function w simultaneously to measure sparsity using the weighted $\mathcal{S}_{q,w}$-measure with guaranteed convergence properties.

COROLLARY 8.16. *Consider the coupling weight* (8.39) *with parameter* $\sigma' = \frac{\sigma}{\eta}$ *for some* $\eta, \delta \in (0,1)$ *and* $\sigma > 0$. *Let us define parameter* $\beta = q\ln\epsilon^{-1}$. *Then for every* $K \in \mathcal{R}_{\sigma,\delta}(\mathbb{Z})$ *we have*

$$\lim_{\epsilon\to 0^+}\;\mathbb{P}\Big\{\;\big|\Psi_{K,e_{\sigma',\delta}}(q,\epsilon)-\gamma'\big| < \epsilon_0\;\Big\}\;=\;1, \tag{8.48}$$

where $\gamma' := (1-\eta^\delta)^{\frac{1}{\delta}}\;\beta^{\frac{1}{\delta}}\;\Gamma\left((1+\delta)/\delta\right)$.

Remark 8.17. Similar results can be obtained for polynomial weight functions. We leave detailed arguments to the interested reader.

There is an inherent algorithm in the result of Theorem 8.15 that provides us with a roadmap to compute a near-ideal sparsification length for a given spatially decaying state feedback controller. Suppose that we are given a subexponentially decaying feedback gain matrix. For a given truncation length $\mathfrak{T} \geq 1$, one can compute a conservative value for parameter ϵ using the following equation:

$$\epsilon = e^{-(\mathfrak{T}/\sigma)^\delta}. \tag{8.49}$$

By fixing the value of parameter β, one can compute a corresponding exponent q using the value of ϵ from (8.49) as well as parameter γ. Knowing all these parameters enables us to compute the value of the sparsity indicator function using its definition in (8.40). An iterative procedure can be used to obtain the corresponding values of the sparsity indicator function for all values of $\mathfrak{T} \geq 1$ by repeating the above-mentioned steps. Let us denote by $\mathfrak{T}_{\text{near-ideal}}$ the value of the sparsification length beyond which the value of the sparsity indicator function converges to a number close to γ with an acceptable error bound. The existence of such a sparsification length for a given state feedback controller implies that there exists an exponent $0 < q_{\text{near-ideal}} < 1$ such that the sparsity indicator function of the controller converges to a number close to one for all $0 < q < q_{\text{near-ideal}}$. According to fundamental limit (8.33) and the inherent tradeoff between truncation length and performance loss, one can select a near-optimal sparsification length for a given state feedback controller that has a desirable level of performance loss and satisfy the following inequality:

$$\mathfrak{T}_{\text{near-optimal}} \geq \max\big\{\mathfrak{T}_s, \mathfrak{T}_{\text{near-ideal}}\big\}, \tag{8.50}$$

where $\mathfrak{T}_s$ is given by (8.33).

8.8. Simulation results. In general, calculating a closed-form solution for the LQR problem for infinite-dimensional systems with no specific symmetry is usually impossible. Therefore, in order to demonstrate our results through simulations we consider a spatially invariant system, where finding a closed-form solution is possible. For general spatially decaying systems, one should use finite-section methods to compute degrees of sparsity and spatial locality; see our discussion in section 9.3 for more details. We consider the heat equation on an infinite bar with distributed heat injection $v(s,t)$ as its control input:

$$\frac{\partial}{\partial t}\psi(s,t) - \kappa\,\frac{\partial^2}{\partial s^2}\psi(s,t) = v(s,t), \tag{8.51}$$

where it is assumed that the thermal diffusivity coefficient is constant and $\kappa = 1$. This problem was studied in [4]. Using the result of Example 8.7, we can discretize (8.51) and get the following state-space model:

$$\dot{x}(t) = Ax(t) + u(t), \tag{8.52}$$

where $x(t) = [\ \ldots, \psi((i-1)h,t), \psi(ih,t), \psi((i+1)h,t), \ldots\ ]^*$ is the state variable and $u(t) = [\ \ldots, v((i-1)h,t), v(ih,t), v((i+1)h,t), \ldots\ ]^*$ is the control input. The state-space matrix A is the state-space matrix in autonomous system (8.52) with a normalized parameter $h = 1$. The resulting system is a spatially invariant system. Let us consider the LQR problem in section 3 for this system with weight matrices $Q = R = I$. The solution to the Riccati equation (7.1) can be obtained using a spatial Fourier transform (see [4] for more details). The Fourier transform of all matrices is

$$\hat{A}(\omega) = -2 + e^{i\omega} + e^{-i\omega},\ \hat{B}(\omega) = 1,\ \hat{Q}(\omega) = 1,\ \hat{R}(\omega) = 1. \tag{8.53}$$

In this case, $K = -X$, where X is the unique solution of the Riccati equation with Fourier transform

$$\hat{X}(\omega) = 2 - e^{i\omega} - e^{-i\omega} + \sqrt{(2 - e^{i\omega} - e^{-i\omega})^2 + 1} = \sum_{k=-\infty}^{\infty} X_k\, e^{-ik\omega},$$

where the Fourier coefficients for $k \geq 0$ can be calculated as follows:

$$\begin{aligned} X_k &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ik\omega}}{-2 + e^{i\omega} + e^{-i\omega} + \sqrt{(-2 + e^{i\omega} + e^{-i\omega})^2 + 1}}\,d\omega \\ &= \frac{1}{2\pi i}\int_{\Upsilon} \frac{z^k}{(z-1)^2 + \sqrt{(z-1)^2 + z^2}}\,dz = \sigma^* z_0^k + \sigma z_0^{-k}, \end{aligned} \tag{8.54}$$

where $z_0 = 0.5804 + 0.6063i$, $\sigma = 0.2822 + 0.3593i$, and Υ is the unit circle. We get

$$X_k = 0.9137\,\cos(0.9050 - 0.7636k)\, e^{-0.1752k}.$$

Since the solution of the Riccati equation is self-adjoint, it follows that $X_k = X_{-k}$ for all $k \in \mathbb{Z}$. Thus, the LQR solution $K = [K_{ij}]_{i,j\in\mathbb{Z}}$ can be expressed by

$$K_{ij} = a_0\,\cos(a_1 - a_2|i-j|)\, e^{-|i-j|/\sigma'}, \tag{8.55}$$

in which $a_0 = -0.9137$, $a_1 = 0.9050$, $a_2 = 0.7636$, and $\sigma' = 5.7078$. The feedback gain K is exponentially decaying; therefore, it is also subexponentially decaying and

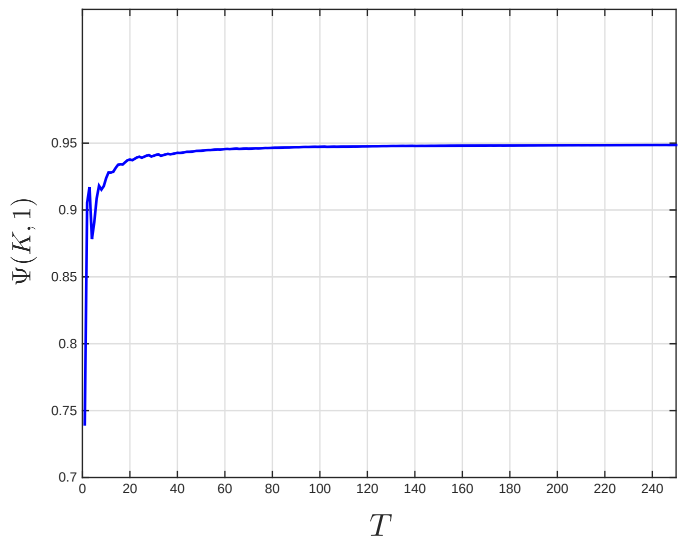


FIG. 3. *The sparsity indicator function* (8.40) *versus range of truncation length* $\mathfrak{T}$.

belongs to $\mathcal{S}_{q,e_{\sigma,\delta}}$ for all $\delta \in [0,1)$ and $\sigma > 0$. Our goal is to apply results of Theorem 8.15 in section 8.7 and compute a near-optimal sparsification length for feedback gain K. Since K is spatially invariant, it follows that

$$\|K\|^q_{\mathcal{S}_{q,w_0}} = |a_0|^q \left(1 + 2\sum_{k=1}^{\infty} |\cos(a_1 - a_2 k)|^q \, e^{-qk/\sigma'}\right).$$

Therefore, in our simulations we may use an $N \times N$ spatial truncation of K in order to compute the $\mathcal{S}_{q,w_0}$-measure of K. It is assumed that $N = 500$ and the truncated feedback is obtained by $K_N = [K_{ij}]_{1 \le i,j \le N}$. We compute the value of the sparsity indicator function (8.40) for K_N as a function of truncation length $1 \le \mathfrak{T} \le \frac{N}{2}$. For a given truncation length and parameter $\beta = 0.1054$, we apply (8.49) for $\delta = 0.98$ and $\sigma = \sigma'$ along with $\beta = q \ln \epsilon^{-1}$ to compute the value of the corresponding q that can then be used to calculate $\Psi_{K,w_0}(q,\epsilon)$ according to (8.40). Our simulation results in Figure 3 indicate that $\mathfrak{T}_{\text{near-ideal}} \approx 24$ is reasonable. The minimum stabilizing truncation length for this system is $\mathfrak{T}_s = 1$, because A is negative definite and $K^{\mathfrak{T}}$ is strictly negative definite for all $\mathfrak{T} \ge 1$. Thus, it follows that $\mathfrak{T}_{\text{near-optimal}} \gtrsim 24$. The value of the corresponding $q_{\text{near-ideal}}$ is 0.0258. According to Theorem 8.15, the value of the $\mathcal{S}_{q,w_0}$-measure of K is a viable sparsity measure in probability for all $0 < q < q_{\text{near-ideal}}$.

9. Conclusion and discussion.

9.1. Generalization to all norms. The focus of this paper is on the class of spatially decaying matrices with exponents $0 < q \le 1$. Nevertheless, our results hold for all exponents $0 < q \le \infty$. With some slight modifications, our method of analysis and all results can be extended to cover the range of exponents $1 \le q \le \infty$.

9.2. Finite-dimensional spatially distributed systems. One can lift a finite-dimensional spatially decaying system to an infinite-dimensional spatially decaying system by defining an extension of a finite-dimensional matrix to an infinite-dimensional matrix. Suppose that $A, B, C, D \in \mathbb{R}^{n\times n}$ are state-space matrices of a linear time-invariant system and $Q, R \in \mathbb{R}^{n\times n}$ are the weight matrices in the LQR problem. The infinite-dimensional extension of a matrix can be defined using the following operation:

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad Z \longmapsto \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} \quad \text{for } Z = B, C, D, Q, R,$$

where I and 0 are the identity and zero operators in $\mathcal{B}(\ell^2(\mathbb{G}))$. The lifted operators have specific block diagonal structures which enable us to show that the unique solution of the corresponding Lyapunov equation (6.1) takes the following block diagonal form:

$$\begin{bmatrix} P & 0 \\ 0 & \frac{1}{2}I \end{bmatrix}, \tag{9.1}$$

where P is the solution of the corresponding finite-dimensional Lyapunov equation. The unique solution of the corresponding algebraic Riccati equation (7.1) admits the following block diagonal form:

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \tag{9.2}$$

where X is the unique solution of the corresponding finite-dimensional Riccati equation. Therefore, the results of sections 6 and 7 can be applied to the lifted infinite-dimensional linear systems. Due to the specific block structures of solutions (9.1) and (9.2), one may obtain accurate spatial decay rates for these solutions based on our present results, which are independent of the system's dimension n. Our extensive simulation results suggest that similar sparsity and spatial localization properties should hold for finite-dimensional spatially decaying systems.

9.3. Spatial truncation and performance bounds. Our proposed methodology in this paper provides quantitative measures to determine the degree of sparsity and spatial localization for a large class of spatially decaying systems and their LQR feedback controllers. This information necessitates the development of algorithmic methods to construct localized models with quantitative estimates, localized feedback control laws with quantitative performance bounds, and feedback control laws with sparse information structures. In section 7, we showed that for a given LQR feedback controller there exists a class of spatially localized feedback controllers that can be calculated by direct spatial truncation of the LQR solution. One remaining important problem is to investigate the asymptotic behavior of the performance index $\Pi_K(\mathfrak{T}, x_0)$ as $\mathfrak{T}$ tends to infinity. Moreover, it is an interesting and open idea to apply finite-section approximation techniques [22] in order to reduce the control design complexity and quantify the convergence rate of the finite-section approximations for spatially decaying systems.

9.4. Regularized optimal control formulations to enhance sparsity and spatial localization. The inherent spatial decay property of the solution of an LQR problem for spatially decaying systems suggests that searching for sparse linear quadratic state feedback controllers should naturally have lower computational complexity compared to general spatially distributed systems. Our analysis in section 8.7 provides a pathway to quantify what degrees of sparsity and spatial localization one should expect by solving $\mathcal{S}_{0,1}/\mathcal{S}_{q,w}$-regularization methods (for $0 < q \leq 1$) in order to design near-optimal sparse state feedback controllers. The $\mathcal{S}_{q,w}$-regularized optimal control problems (for $0 < q < 1$) are much more broadly applicable with respect to ℓ^1-based relaxations, which results in $\mathcal{S}_{1,w}$-regularized optimal control problems. The reason is that the Gröchenig–Schur class of matrices enjoys the following fundamental inclusion property:

$$\mathcal{S}_{q_1,w}(\mathbb{G}) \subset \mathcal{S}_{q_2,w}(\mathbb{G}) \quad \text{for all } 0 < q_2 \leq q_1 \leq \infty.$$

This implies that the space of spatially decaying systems over $\mathcal{S}_{q,w}(\mathbb{G})$ for $0 < q < 1$ is larger than the space of spatially decaying systems over $\mathcal{S}_{1,w}(\mathbb{G})$, which was originally studied in [31, 32].

9.5. Receding horizon control with state and input constraints. The machinery developed in this paper can be used to analyze the spatial structure of a broader range of optimal control problems, such as constrained finite horizon control (also known as model predictive control). In [31], the problem of constrained finite horizon control for the class of discrete-time spatially decaying systems over $\mathcal{S}_{q,w}(\mathbb{G})$ was considered. The results of this paper along with the techniques developed in [31] can be applied to study sparsity and localization properties of the class of multiparametric quadratic programming problems over q-Banach algebras.

9.6. Systems governed by partial differential equations. A natural generalization of the present results is to study spatial localization of feedback control laws for linear systems that are defined using differential operators/pseudodifferential operators/integral operators. The LQR problem for this class of systems involves state-space and weight matrices A, B, Q, R whose elements are differential operators/pseudodifferential operators/integral operators. A special class of such systems includes translation invariant operators, such as partial differential operators with constant coefficients, spatial shift operators, spatial convolution operators, or a linear combination of several of such operators [23], or general pseudodifferential operator and integral operators [17, 45]. In general, some of the operators may be unbounded, so the notion of a solution for the partial differential equation system requires some deep mathematics, such as theory of operator semigroups, theory of distributions, and theory of Fourier integral operators [14, 16]. For translation invariant systems, the feedback control law has the spatial convolution form. The main idea is to exploit the spatial invariance property of such systems by taking a spatial Fourier transform and study the diagonalized form (in the Fourier domain) of the corresponding LQR problem [4, 13]. The Fourier approach is only applicable to the class of spatially invariant systems. To generalize the present results, one would need to develop suitable operator algebras in order to quantify the sparsity and spatial localization for the class of partial differential equation systems, and this is beyond the scope of this paper and will be discussed in our future works.

9.7. Nonlinear spatially decaying systems. The notion of spatial decay can be extended to study spatially distributed systems with nonlinear dynamics. Let us consider an infinite-dimensional nonlinear system of the form

$$\dot{x} \;=\; f(x, u), \tag{9.3}$$

where $x, u \in \ell^2(\mathbb{G})$ and $f : \ell^2(\mathbb{G}) \times \ell^2(\mathbb{G}) \to \ell^2(\mathbb{G})$. It is assumed that a suitable notion of a solution exists for this system. We say that system (9.3) is spatially decaying if f has continuous bounded gradients $\nabla_x f$ (with respect to x) and $\nabla_u f$ (with respect to u) on a q-Banach algebra for $0 < q \leq \infty$. This is a natural extension in order to develop a theory for analysis of nonlinear spatially distributed systems including quantification of the sparsity and spatial localization measures for nonlinear networks and approximation techniques. As part of this theory, one of the fundamental issues to address is the inverse-closedness property of nonlinear maps. In [40], it is shown that under some conditions the inverse-closedness property holds for some class of nonlinear functionals over some inverse-closed Banach algebras.

10. Appendix on inverse-closed subalgebras.

10.1. Proof of Proposition 8.3. The class of subexponentially decaying coupling weights is the most suitable class of coupling weights to study sparsity features of spatially decaying matrices. For the subexponential coupling weight function $e_{\sigma,\delta}$, one can conclude from the inequality

$$(1+t)^\delta \;\le\; 1+(2^\delta-1)t^\delta \;\le\; 1+t^\delta \quad \text{for all } t\in[0,1]$$

that $e_{\sigma,\delta}$ is submultiplicative and that another subexponential weight $e_{\sigma',\delta}$ with $\sigma' = \sigma/\sqrt[\delta]{(2^\delta-1)}$ can be adopted as its companion weight [42, Example A.3]. Furthermore, the second inequality in Definition 8.2 holds with constants D_e and θ_e, because

$$\begin{aligned}
&\inf_{\tau\ge 0}\left\{\left(\sum_{|i-j|<\tau}\big(e_{\sigma',\delta}(i-j)\big)^{\frac{2q}{2-q}}\right)^{1-\frac{q}{2}} + t\sup_{|i-j|\ge\tau}\left(\frac{e_{\sigma',\delta}(i-j)}{e_{\sigma,\delta}(i-j)}\right)^{q}\right\}\\
&\le\;\inf_{\tau\ge 0}\left\{\exp\big((2^\delta-1)q\sigma^{-\delta}\tau^\delta\big)\left(\sum_{|i-j|<\tau}1\right)^{1-\frac{q}{2}} \;+\; t\exp\big(-(2-2^\delta)q\sigma^{-\delta}\tau^\delta\big)\right\}\\
&\le 2^{d+1-\frac{dq}{2}}\; t^{1-\frac{q\delta(2-2^\delta)}{q\delta+d(1-\frac{q}{2})\sigma^\delta}}.
\end{aligned}$$

Next, we consider the class of polynomial coupling weights. This class of coupling weights is submultiplicative: $\pi_{\alpha,\sigma}(i,j)\le\pi_{\alpha,\sigma}(i,k)\pi_{\alpha,\sigma}(k,j)$ for all $i,j,k\in\mathbb{Z}^d$. Also it is straightforward to verify that

$$\pi_{\alpha,\sigma}(i,j)\le 2^\alpha\big(\pi_{\alpha,\sigma}(i,k)\;+\;\pi_{\alpha,\sigma}(k,j)\big)\quad\text{for all } i,j,k\in\mathbb{Z}^d.$$

Thus, the constant weight $u_{\alpha,\sigma} := \big[u_{\alpha,\sigma}(i,j)\big]_{i,j\in\mathbb{Z}^d}$ with $u_{\alpha,\sigma}(i,j)=2^\alpha$ can be used as a companion weight of the polynomial coupling weight $\pi_{\alpha,\sigma}$. Moreover, the second inequality in Definition 8.2 holds with constants D_π and θ_π, because

$$\begin{aligned}
&\inf_{\tau\ge 0}\left\{\left(\sum_{|i-j|<\tau}\big(u_\alpha(i,j)\big)^{\frac{2q}{2-q}}\right)^{1-\frac{q}{2}} \;+\; t\sup_{|i-j|\ge\tau}\left(\frac{u(i,j)}{w(i,j)}\right)^{q}\right\}\\
&\le 2^{q\alpha}\inf_{\tau\ge 0}\left\{\left(\sum_{|i-j|<\tau}1\right)^{1-\frac{q}{2}} \;+\; t\;\left(1+\frac{\tau}{\sigma}\right)^{-q\alpha}\right\}\\
&\le 2^{q\alpha+1}\max\left\{1,(2\sigma)^{d(1-\frac{q}{2})}\right\}\,t^{1-\frac{q\alpha}{q\alpha+d(1-\frac{q}{2})}}.
\end{aligned}$$

10.2. Weak submultiplicative property resulting from Definition 8.2. Suppose that w is a weight that satisfies the inequalities in Definition 8.2. For $0<$

$q \le 1$, we get

$$\sup_{j\in\mathbb{G}}\left(\frac{u(i,j)}{w(i,j)}\right)^q \le \inf_{\tau\ge 0}\left(\sup_{\substack{j\in\mathbb{G}\\ \rho(i,j)<\tau}}\left(\frac{u(i,j)}{w(i,j)}\right)^q + \sup_{\substack{j\in\mathbb{G}\\ \rho(i,j)\ge\tau}}\left(\frac{u(i,j)}{w(i,j)}\right)^q\right)$$
$$\le \inf_{\tau\ge 0}\left(\Big(\sum_{\substack{j\in\mathbb{G}\\ \rho(i,j)<\tau}}|u(i,j)|^{\frac{2q}{2-q}}\Big)^{1-\frac{q}{2}} + \sup_{\substack{j\in\mathbb{G}\\ \rho(i,j)\ge\tau}}\left(\frac{u(i,j)}{w(i,j)}\right)^q\right) \le D \quad \text{for all } i\in\mathbb{G}.$$

The second inequality holds as $w(i,j)\ge 1$ for $i,j\in\mathbb{G}$, and the last inequality follows by applying the second inequality in Definition 8.2 with $t=1$. The above estimate for the weight w and its companion weight u along with property (8.2) prove (8.5), a weak version of the submultiplicative property (8.1). We refer the reader to [41, 42] for similar arguments for $1\le q\le\infty$.

10.3. Proof of Theorem 8.9. According to the definition of $\mathcal{S}_{0,1}$-measure, there exist i_0 and $j_0\in\mathbb{G}$ such that

$$\|A\|_{\mathcal{S}_{0,1}(\mathbb{G})} = \max\big\{\|a_{i_0\cdot}\|_{\ell^0(\mathbb{G})}, \|a_{\cdot j_0}\|_{\ell^0(\mathbb{G})}\big\}. \tag{10.1}$$

Then from (8.22) and (10.1) it follows that

$$\lim_{q\to 0}\max\left\{\sum_{j\in\mathbb{G}}|a_{i_0j}|^q w(i_0,j)^q, \sum_{i\in\mathbb{G}}|a_{ij_0}|^q w(i,j_0)^q\right\} \;=\; \|A\|_{\mathcal{S}_{0,1}(\mathbb{G})}. \tag{10.2}$$

One observes that
(10.3)
$$\max\left\{\sum_{j\in\mathbb{G}}|a_{i_0j}|^q w(i_0,j)^q, \sum_{i\in\mathbb{G}}|a_{ij_0}|^q w(i,j_0)^q\right\} \le\; \|A\|^q_{\mathcal{S}_{q,w}(\mathbb{G})} \le M^q\|A\|_{\mathcal{S}_{0,1}(\mathbb{G})},$$

where $M=\sup_{i,j\in\mathbb{G}}|a_{ij}|w(i,j)<\infty$ by the assumption on bounded entries. Letting $q\to 0$ in (10.3) and applying (10.2), we establish the limit (8.24).

10.4. Proof of Theorem 8.5. First, we show that $\mathcal{S}_{q,w}(\mathbb{G})$ is a q-Banach algebra. It is straightforward to verify properties (i) and (ii) in Definition 4.1. The q-subadditivity of the $\mathcal{S}_{q,w}$-measure follows from the inequality $(\alpha+\beta)^q\le\alpha^q+\beta^q$ for all $\alpha,\beta\ge 0$. Hence the $\mathcal{S}_{q,w}$-measure satisfies property (iii) in Definition 4.1. The submultiplicativity of the q-norm holds as

$$\|z\|_1 \le \|z\|_q, \quad z\in\ell^q(\mathbb{G}), \tag{10.4}$$

where $0<q\le 1$. This shows that $\mathcal{S}_{q,w}$ has property (iv) in Definition 4.2.

In the next step, we prove that $\mathcal{S}_{q,w}(\mathbb{G})$ is a proper q-Banach algebra. It is straightforward to verify property (P1). Applying (10.4) shows that $\mathcal{S}_{q,w}(\mathbb{G})$ is continuously imbedded in $\mathcal{S}_{1,w_0}(\mathbb{G})$, where w_0 is the trivial weight on $\mathbb{G}\times\mathbb{G}$ with all entries taking value one. Thus, property (P2) holds. Finally, we verify the differential norm property (P3) for $\mathcal{S}_{q,w}(\mathbb{G})$. For $q=1$, $\mathcal{S}_{q,w}(\mathbb{G})$ has the differential norm property; we refer the reader to [42] for a proof. Therefore, from now on we suppose that $0<q<1$. For every $A=[a_{ij}]_{i,j\in\mathbb{G}}$ and $B=[b_{ij}]_{i,j\in\mathbb{G}}$ in $\mathcal{S}_{q,w}(\mathbb{G})$, let us denote $C=AB$, where $C=[c_{ij}]_{i,j\in\mathbb{G}}$. From (8.2), it follows that

$$\sup_{i\in\mathbb{G}}\sum_{j\in\mathbb{G}}|c_{ij}|^q w(i,j)^q \le \|A\|^q_{\mathcal{S}_{q,w}}\sup_{k\in\mathbb{G}}\sum_{j\in\mathbb{G}}|b_{kj}|^q u(k,j)^q + \|B\|^q_{\mathcal{S}_{q,w}}\sup_{i\in\mathbb{G}}\sum_{k\in\mathbb{G}}|a_{ik}|^q u(i,k)^q.$$

In the following, we obtain upper bounds for each term in the right-hand side of the inequality above. Let us first consider

(10.5)

$$\begin{aligned}
\sup_{k\in\mathbb{G}} \sum_{j\in\mathbb{G}} |b_{kj}|^q u(k,j)^q &= \sup_{k\in\mathbb{G}} \inf_{\tau\geq 0} \Bigg(\sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)<\tau}} + \sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)\geq\tau}} \Bigg) |b_{kj}|^q \, u(k,j)^q \\
&\leq \sup_{k\in\mathbb{G}} \inf_{\tau\geq 0} \Bigg\{ \Bigg(\sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)<\tau}} |b_{kj}|^2 \Bigg)^{\frac{q}{2}} \Bigg(\sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)<\tau}} |u(k,j)|^{\frac{2q}{2-q}} \Bigg)^{1-\frac{q}{2}} + \|B\|^q_{\mathcal{S}_{q,w}} \sup_{\substack{j\in\mathbb{G}\\ \rho(k,j)\geq\tau}} \left(\frac{u(k,j)}{w(k,j)}\right)^q \Bigg\} \\
&\leq \sup_{k\in\mathbb{G}} \inf_{\tau\geq 0} \Bigg\{ \|B\|^q_{\mathcal{B}(\ell^2)} \Bigg(\sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)<\tau}} |u(k,j)|^{\frac{2q}{2-q}} \Bigg)^{1-\frac{q}{2}} + \|B\|^q_{\mathcal{S}_{q,w}} \sup_{\substack{j\in\mathbb{G}\\ \rho(k,j)\geq\tau}} \left(\frac{u(k,j)}{w(k,j)}\right)^q \Bigg\} \\
&\leq D \, \|B\|^{q\theta}_{\mathcal{B}(\ell^2)} \, \|B\|^{q(1-\theta)}_{\mathcal{S}_{q,w}}.
\end{aligned}$$

In the last two inequalities, we apply Hölder's inequality, inequalities in Definition 8.2 with $t = \frac{\|B\|^q_{\mathcal{S}_{q,w}}}{\|B\|^q_{\mathcal{B}(\ell^2)}} \geq 1$, and the following inequality:

$$\sum_{\substack{j\in\mathbb{G}\\ \rho(k,j)<\tau}} |b_{kj}|^2 \;\leq\; \sum_{j\in\mathbb{G}} |b_{kj}|^2 = \|B^* e_k\|_2^2 \leq \|B^*\|^2_{\mathcal{B}(\ell^2)} = \|B\|^2_{\mathcal{B}(\ell^2)},$$

where $e_k = [\delta_{kj}]_{j\in\mathbb{G}}$ is the kth standard Euclidean basis. In a similar manner, we can show that

$$\sup_{i\in\mathbb{G}} \sum_{k\in\mathbb{G}} |a_{ik}|^q u(i,k) \leq D \|A\|^{q\theta}_{\mathcal{B}(\ell^2)} \, \|A\|^{q(1-\theta)}_{\mathcal{S}_{q,w}}. \tag{10.6}$$

By combining inequalities (10.5) and (10.6), we get

$$\sup_{i\in\mathbb{G}} \sum_{j\in\mathbb{G}} |c_{ij}|^q \, w(i,j)^q \;\leq\; D \, \|A\|^q_{\mathcal{S}_{q,w}} \|B\|^q_{\mathcal{S}_{q,w}} \left(\left(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{S}_{q,w}}}\right)^{q\theta} + \left(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{S}_{q,w}}}\right)^{q\theta} \right).$$

Using a similar argument along with symmetry of a weight, we get

$$\sup_{j\in\mathbb{G}} \sum_{i\in\mathbb{G}} |c_{ij}|^q \, w(i,j)^q \;\leq\; D \, \|A\|^q_{\mathcal{S}_{q,w}} \|B\|^q_{\mathcal{S}_{q,w}} \left(\left(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{S}_{q,w}}}\right)^{q\theta} + \left(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{S}_{q,w}}}\right)^{q\theta} \right).$$

Therefore, we can conclude that $\mathcal{S}_{q,w}(\mathbb{G})$ has the differential norm property (P3).

10.5. Proof of Theorem 4.6. For every $n \geq 1$, let us write $n = \sum_{j=0}^N \epsilon_j 2^j$ with $\epsilon_N = 1$ and $\epsilon_j \in \{0,1\}$ for $0 \leq j \leq N-1$. For a given $A \in \mathcal{A}$, by performing induction on the differential norm property we obtain

$$\begin{aligned}
\|A^n\|^q_{\mathcal{A}} &\leq K_0^{q\epsilon_0} \|A\|^{q\epsilon_0}_{\mathcal{A}} \|A^{n-\epsilon_0}\|^q_{\mathcal{A}} \leq \cdots \\
&\leq (2D)^{\sum_{j=0}^{N-1}(2-\theta)^j} \, K_0^{q\sum_{j=0}^{N-1}\epsilon_j(2-\theta)^j} \left(\frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{q\sum_{j=0}^N \epsilon_j(2-\theta)^j} \|A\|^{qn}_{\mathcal{B}(\ell^2)}
\end{aligned}$$

$$\leq \begin{cases} \left(2DK_0^q\right)^{(1-\theta)^{-1} n^{\log_2(2-\theta)}} \|A\|^{qn}_{\mathcal{B}(\ell^2)} \left(\frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{q\frac{2-\theta}{1-\theta} n^{\log_2(2-\theta)}} & \text{if} \quad 0<\theta<1, \\[2ex] \left(2DK_0^q\right)^{\log_2 n} \left(\frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{q\log_2 n + q} \|A\|^{qn}_{\mathcal{B}(\ell^2)} & \text{if} \quad \theta = 1, \end{cases} \tag{10.7}$$

where D, K_0, θ are the positive constants in Definitions 4.2–4.4 (cf. [40, 44]). We will use this result in the following proof steps. For a given matrix $A \in \mathcal{A}$ with $A^{-1} \in \mathcal{B}(\ell^2(\mathbb{G}))$, let us define $B = I - \|A\|^{-2}_{\mathcal{B}(\ell^2)} A^*A$. It is straightforward to verify that

$$0 \preceq B \preceq r_0 I, \tag{10.8}$$

where $r_0 = 1 - \big(\|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}(\ell^2)}\big)^{-2} \in [0, 1)$. When $r_0 = 0$, the conclusion follows immediately, as in this case we have $A^{-1} = \|A^{-1}\|^{-2}_{\mathcal{B}(\ell^2)} A^*$. Thus, from now on we assume that $r_0 \in (0, 1)$. We apply properties of $\|\cdot\|_{\mathcal{A}}$ in Definitions 4.1, 4.2, and 4.4, and we get

$$\|B\|^q_{\mathcal{A}} \le \|I\|^q_{\mathcal{A}} + \|A\|^{-2q}_{\mathcal{B}(\ell^2)} \|A^*A\|^q_{\mathcal{A}} \le M^q + K_0^q \|A\|^{-2q}_{\mathcal{B}(\ell^2)} \|A\|^{2q}_{\mathcal{A}}. \tag{10.9}$$

For $\theta \in (0, 1)$, this together with (10.8) and inequality (10.7) leads to

$$\begin{aligned}\|B^n\|^q_{\mathcal{A}} &\le \big(2DK_0^q\big)^{(1-\theta)^{-1} n^{\log_2(2-\theta)}} \|B\|^{qn}_{\mathcal{B}(\ell^2)} \left(\frac{\|B\|_{\mathcal{A}}}{\|B\|_{\mathcal{B}(\ell^2)}}\right)^{q\frac{2-\theta}{1-\theta} n^{\log_2(2-\theta)}} \\ &\le \big(2DK_0^q\big)^{(1-\theta)^{-1} n^{\log_2(2-\theta)}} r_0^{qn} \left(\frac{M^q + K_0^q \|A\|^{-2q}_{\mathcal{B}(\ell^2)} \|A\|^{2q}_{\mathcal{A}}}{r_0^q}\right)^{\frac{2-\theta}{1-\theta} n^{\log_2(2-\theta)}}\end{aligned}$$

for all $n \ge 1$. Similarly, for $\theta = 1$,

$$\|B^n\|^q_{\mathcal{A}} \le \big(2DK_0^q\big)^{\log_2 n} r_0^{qn} \left(\frac{M^q + K_0^q \|A\|^{-2q}_{\mathcal{B}(\ell^2)} \|A\|^{2q}_{\mathcal{A}}}{r_0^q}\right)^{\log_2 n + 1}$$

for $n \ge 1$. Therefore, $I - B$ is invertible in $\mathcal{A}$, and

$$\big\|(I - B)^{-1}\big\|^q_{\mathcal{A}} \le \sum_{n=1}^{\infty} \|B^n\|^q_{\mathcal{A}} < \infty$$

as $r_0 \in (0, 1)$ and $\theta \in (0, 1]$. We recall the matrix identity

$$A^{-1} = (A^*A)^{-1} A^* = \|A\|^{-2}_{\mathcal{B}(\ell^2)} (I - B)^{-1} A^*.$$

From this, we can conclude that A is invertible in $\mathcal{A}$.

10.6. Proof of Theorem 4.7. The inclusion $\sigma_{\mathcal{A}}(A) \subset \sigma_{\mathcal{B}(\ell^2(\mathbb{G}))}(A)$ follows directly from the result of Theorem 4.6, while the reverse inclusion $\sigma_{\mathcal{B}(\ell^2(\mathbb{G}))}(A) \subset \sigma_{\mathcal{A}}(A)$ holds by continuous imbedding of $\mathcal{A}$ in $\mathcal{B}(\ell^2(\mathbb{G}))$.

10.7. *q*-Banach algebras of block matrices. Given a proper q-Banach subalgebra $\mathcal{A}$ of $\mathcal{B}(\ell^2(\mathbb{G}))$, define an algebra of $n \times n$ matrices with entries in $\mathcal{A}$ by

$$\mathcal{M}_n(\mathcal{A}) = \Big\{ \mathbf{A} := [a_{ij}] \Big| \|\mathbf{A}\|_{\mathcal{M}_n(\mathcal{A})} < \infty \Big\}, \tag{10.10}$$

where

$$\|\mathbf{A}\|_{\mathcal{M}_n(\mathcal{A})} = \left(\sum_{i,j=1}^{n} \|a_{ij}\|^q_{\mathcal{A}} \right)^{1/q}. \tag{10.11}$$

LEMMA 10.1. *For $0 < q \leq 1$, let us assume that $\mathcal{A}$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}))$. Then the algebra $\mathcal{M}_n(\mathcal{A})$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}^n))$, where $\mathbb{G}^n = \underbrace{\mathbb{G} \times \cdots \times \mathbb{G}}_{n}$.*

Proof. By (10.10) and (10.11), the quasi norm $\|\cdot\|_{\mathcal{M}_n(\mathcal{A})}$ satisfies the first three requirements in Definition 4.2 for a q-Banach algebra. Given $\mathbf{A} := [a_{ij}]$ and $\mathbf{B} := [b_{ij}]$ in $\mathcal{M}_n(\mathcal{A})$, one may verify that

$$\|\mathbf{AB}\|^q_{\mathcal{M}_n(\mathcal{A})} \leq \sum_{i,j=1}^{n} \Big\| \sum_{k=1}^{n} a_{ik} b_{kj} \Big\|^q_{\mathcal{A}} \;\leq\; K_0 \; \|\mathbf{A}\|^q_{\mathcal{M}_n(\mathcal{A})} \; \|\mathbf{B}\|^q_{\mathcal{M}_n(\mathcal{A})}, \tag{10.12}$$

where K_0 is the constant in the definition of the q-Banach algebra $\mathcal{A}$. Therefore, $\mathcal{M}_n(\mathcal{A})$ is a q-Banach algebra. The q-Banach algebra $\mathcal{M}_n(\mathcal{A})$ is closed under the complex conjugate operation by (4.1) and (10.11). For $\mathbf{A} := [a_{ij}] \in \mathcal{M}_n(\mathcal{A})$ and $x = [x_1, \dots, x_n]$ with $\|x_1\|^2_{\ell^2(\mathbb{G})} + \cdots + \|x_n\|^2_{\ell^2(\mathbb{G})} \leq 1$,

$$\begin{aligned}\big\|\mathbf{A}x\big\|^2_{\ell^2(\mathbb{G}^n)} &= \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} a_{ij} x_j \Big\|^2_{\ell^2(\mathbb{G})} \;\leq\; \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|a_{ij}\|_{\mathcal{B}(\ell^2(\mathbb{G}))} \|x_j\|_{\ell^2(\mathbb{G})} \right)^2 \\ &\leq \sum_{i,j=1}^{n} \|a_{ij}\|^2_{\mathcal{B}(\ell^2(\mathbb{G}))} \leq \|\mathbf{A}\|^2_{\mathcal{M}_n(\mathcal{A})},\end{aligned}$$

where the last inequality follows from (4.2) and (10.11). Thus, the q-Banach algebra $\mathcal{M}_n(\mathcal{A})$ is a subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}^n))$ and satisfies

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(\mathbb{G}^n))} \leq \|\mathbf{A}\|_{\mathcal{M}_n(\mathcal{A})} \quad \text{for all } \mathbf{A} \in \mathcal{M}_n(\mathcal{A}).$$

The q-Banach algebra $\mathcal{M}_n(\mathcal{A})$ is a differential subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}^n))$ because for any $\mathbf{A} := [a_{ij}]$ and $\mathbf{B} := [b_{ij}]$ in $\mathcal{M}_n(\mathcal{A})$,

$$\begin{aligned}\|\mathbf{AB}\|^q_{\mathcal{M}_n(\mathcal{A})} &\leq D \sum_{i,j,k=1}^{n} \|a_{ik}\|^q_{\mathcal{A}} \|b_{kj}\|^q_{\mathcal{A}} \left(\left(\frac{\|a_{ik}\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}{\|a_{ik}\|_{\mathcal{A}}} \right)^{q\theta} + \left(\frac{\|b_{kj}\|_{\mathcal{B}(\ell^2(\mathbb{G}))}}{\|b_{kj}\|_{\mathcal{A}}} \right)^{q\theta} \right) \\ &\leq nD \; \|\mathbf{A}\|^q_{\mathcal{M}_n(\mathcal{A})} \|\mathbf{B}\|^q_{\mathcal{M}_n(\mathcal{A})} \left(\left(\frac{\|\mathbf{A}\|_{\mathcal{B}(\ell^2(\mathbb{G}^n))}}{\|\mathbf{A}\|_{\mathcal{M}_n(\mathcal{A})}} \right)^{q\theta} + \left(\frac{\|\mathbf{B}\|_{\mathcal{B}(\ell^2(\mathbb{G}^n))}}{\|\mathbf{B}\|_{\mathcal{M}_n(\mathcal{A})}} \right)^{q\theta} \right).\end{aligned}$$

Thus, $\mathcal{M}_n(\mathcal{A})$ is a proper q-Banach subalgebra of $\mathcal{B}(\ell^2(\mathbb{G}^n))$. ∎

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