Automatica 80 (2017) 305–312

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

New spectral bounds on $\mathcal{H}_2$-norm of linear dynamical networks ☆

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ARTICLE INFO

Article history:
Received 10 January 2016
Received in revised form
18 October 2016
Accepted 31 December 2016

Keywords:
Network analysis and control
Fundamental limits
Cyclic dynamical networks
Linear consensus networks
$\mathcal{H}_2$-norm
Algebraic Lyapunov Equations

ABSTRACT

In this paper, we obtain new lower and upper bounds for the $\mathcal{H}_2$-norm of a class of linear time-invariant systems subject to exogenous noise inputs. We show that the $\mathcal{H}_2$-norm, as a performance measure, can be tightly bounded from below and above by some spectral functions of state and output matrices of the system. In order to show the usefulness of our results, we calculate bounds for the $\mathcal{H}_2$-norm of some network models with specific coupling or graph structures, e.g., systems with normal state matrices, linear consensus networks with directed graphs, and cyclic linear networks. As a specific example, the $\mathcal{H}_2$-norm of a linear consensus network over a directed cycle graph is computed and shown how its performance scales with the network size. Our proposed spectral bounds reveal the important role and contribution of fast and slow dynamic modes of a system in the best and worst achievable performance bounds under white noise excitation. Finally, we use several numerical simulations to show the superiority of our bounds over the existing bounds in the literature.

1. Introduction

The performance analysis of noisy linear systems has been the focus of numerous studies over the past few decades (Bamieh & Dahleh, 2003; Bamieh, Jovanović, Mitra, & Patterson, 2012; Chandra, Buzi, & Doyle, 2011; Doyle, Glover, Khargonekar, & Francis, 1989; Siami & Motee, 2013a) and the references therein. In a majority of these works, quantifying the corresponding performance measures reduces to solving some Algebraic Lyapunov Equations (ALEs). Although there are several efficient methods to compute the exact solutions of ALEs, their computational complexity increases rapidly when dealing with linear systems with large dimensions. Thus, such algorithms are only applicable to systems of moderate size (Benner, Li, & Penzl, 2008). There are some methods to estimate bounds on the solutions of ALEs (Komaroff, 1988; Kwon, Youn, & Bien, 1985; Lee, 1997; Mori, Fukuma, & Kuwahara, 1987; Wang, Kuo, & Hsu, 1986). Bounds on the solution of an ALE can be used as an approximation of its exact solution, especially for large-scale linear networks as these bounds can usually be calculated through inexpensive computations. In Kwon, Moon, and Ahn (1996), authors summarize some of the previous results up to that date.

The $\mathcal{H}_2$-norm of a noisy linear time-invariant system, from its noise input to the output, has been considered as a viable performance measure in the literature (Bamieh & Dahleh, 2003; Doyle et al., 1989; Siami & Motee, 2013a). This performance measure can be calculated using the solution of an ALE. In this paper, we derive explicit lower and upper bounds for this performance measure. Our proposed bounds are spectral functions of state and output matrices of the system. Furthermore, our proposed bounds are utilized to quantify bounds on the $\mathcal{H}_2$-norm squared of some network models with specific dynamical structures, e.g., systems with normal state matrices, linear consensus networks with directed graphs, and cyclic linear networks with negative feedback. As an important application, our results are applied to a general class of linear consensus networks over directed graphs. Most recent works (Bamieh et al., 2012; Siami & Motee, 2015) investigate the performance of noisy linear consensus networks over undirected graphs. We prove that our performance bounds are tight if the underlying directed graph of the networks is strongly connected and balanced. Moreover, we apply our results to a class of cyclic networks with asymmetric structures. These networks has been used to model certain biochemical pathways (Kholodenko, 2000). We particularly show how the $\mathcal{H}_2$-norm of a cyclic linear dynamical network scales with the network size. It is shown that when all subsystems are identical, the network attains the best achievable performance among all cyclic networks with the same secant criterion. Finally, we compare our proposed bounds to all existing bounds in the literature

☆ This research is supported by NSF-ECCS-1202517, AFOSR-YIP FA9550-13-1-0158, ONR N00014-13-1-0636, and NSF CAREER ECCS-1454022. The material in this paper was partially presented at the 2013 Conference on Control and Its Applications, July 8–10, 2013, San Diego, CA, USA. This paper was recommended for publication in revised form by Associate Editor Yoshio Ebihara under the direction of Editor Richard Middleton.

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http://dx.doi.org/10.1016/j.automatica.2017.01.043
0005-1098/

and use some numerical simulations to show that our bounds are tighter than all previously reported bounds in Fang, Loparo, and Feng (1997), Kwon et al. (1996) and Lee (1997).

2. Mathematical notations

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbf{Re}\{.\}$ denotes the real part of a complex number, $(.)^{\mathrm{T}}$ denotes transpose and $(.)^{\mathrm{H}}$ denotes Hermitian transpose. Matrix $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and matrix $\mathbf{0}$ is the matrix of all zeros. The $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n$ and the centering matrix is defined by $M_n := I_n - \frac{\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}}{n}$. For a square matrix A, $\mathbf{Tr}(A)$ refers to the summation of on-diagonal elements of A. We write $\lambda_{\max}(M)$ (resp., $\lambda_{\min}(M)$) for the maximum (resp., minimum) eigenvalue of M, $\mathbf{diag}[v]$ for a square diagonal matrix with the elements of vector v on its diagonal and zero everywhere else, and $\|\cdot\|_2$ for the 2-norm of a vector. The eigenvalues of a matrix $X \in \mathbb{R}^{n\times n}$ are indexed according to their real-parts in ascending order, i.e., $\mathbf{Re}\{\lambda_1(X)\} \le \mathbf{Re}\{\lambda_2(X)\} \le \cdots \le \mathbf{Re}\{\lambda_n(X)\}$. $\mathbf{E}[v]$ stands for the expected value of random variable v. We employ the big omega notation in order to generalize the concept of asymptotic lower bound in the same way as $\mathcal{O}$ generalizes the concept of asymptotic upper bound. We adopt the following definition according to (Knuth, 1976):

$$f(n) = \Omega(g(n)) \Leftrightarrow g(n) = \mathcal{O}(f(n)), \tag{1}$$

where $\mathcal{O}$ represents the big O notation. In the left hand side of (1), the Ω notation implies that $f(n)$ grows at least of the order of $g(n)$.

3. $\mathcal{H}_2$-norm of noisy linear systems

The steady-state variance of outputs of linear systems driven by external stochastic disturbances can be regarded as a measure of performance. We consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + \xi(t), \tag{2}$$
$$y(t) = Cx(t), \tag{3}$$

with $x(0) = \mathbf{0}$, where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ is the output of the system. For all linear systems in this paper, it is assumed that the input signal $\xi \in \mathbb{R}^n$ is a white noise process with zero mean and identity covariance, i.e.,

$$\mathbf{E}\left[\xi(t)\xi^{\mathrm{T}}(\tau)\right] = I_n\delta(t-\tau), \tag{4}$$

where $\delta(.)$ is the delta function. It is assumed that the state matrix A is Hurwitz.

Definition 1. The $\mathcal{H}_2$-norm of linear system (2)–(3) from ξ to y is defined as the square root of the following quantity

$$\rho_{ss}(A;Q) := \lim_{t\to\infty} \mathbf{E}\left[x^{\mathrm{T}}(t)Qx(t)\right], \tag{5}$$

where $Q = C^{\mathrm{T}}C$.

For unstable linear systems, the outputs of the system have finite steady-state variance as along as the unstable modes of the system are not observable from the output of the system. The value of performance measure (5) for (2)–(3) can be quantified as

$$\rho_{ss}(A;Q) = \mathbf{Tr}(P), \tag{6}$$

where P is the unique solution of the following ALE

$$PA + A^{\mathrm{T}}P + Q = \mathbf{0}. \tag{7}$$

4. New spectral bounds on the $\mathcal{H}_2$-norm

For simplicity of our representations, we present our results for the performance measure (5), instead of the $\mathcal{H}_2$-norm. According to Definition 1 by taking a simple square root, all results can be converted to bounds for the $\mathcal{H}_2$-norm.

Theorem 2. *Suppose that linear system (2)–(3) is stable with input noise covariance (4) and $Q = I_n$. Then, we have*

$$\sum_{i=1}^{n} \frac{-1}{2\mathbf{Re}\{\lambda_i(A)\}} \le \rho_{ss}(A;Q). \tag{8}$$

The lower bound in (8) is achieved if and only if A is normal, i.e., $A^{\mathrm{T}}A = AA^{\mathrm{T}}$. In addition, if the symmetric part of the state matrix A, defined by $A_s := (A^{\mathrm{T}} + A)/2$, is Hurwitz, then we get

$$\rho_{ss}(A;Q) \le \sum_{i=1}^{n} \frac{-1}{2\lambda_i(A_s)}. \tag{9}$$

Proof. Since A is Hurwitz, all its eigenvalues have strictly negative real parts. Therefore, the unique solution of $A^{\mathrm{T}}P + PA + I_n = \mathbf{0}$, can be expressed in the following closed form

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}e^{At}\,dt. \tag{10}$$

According to Schur decomposition theorem (Horn & Johnson, 1990), there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that $A = V(\Gamma + N)V^{\mathrm{H}}$ where $\Gamma = \mathbf{diag}[\lambda_1(A),\ldots,\lambda_n(A)]$, N is strictly upper triangular, and V^{H} is the conjugate transpose of V. Let us consider the integrand of (10)

$$\begin{aligned}\mathbf{Tr}(e^{A^{\mathrm{T}}t}e^{At}) &= \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t}V^{\mathrm{H}}Ve^{(\Gamma+N)t}V^{\mathrm{H}}V)\\ &= \mathbf{Tr}(V^{\mathrm{H}}Ve^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t}e^{(\Gamma+N)t})\\ &= \mathbf{Tr}(Ve^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t}e^{(\Gamma+N)t}V^{\mathrm{H}}).\end{aligned} \tag{11}$$

Furthermore, we have

$$e^{(\Gamma+N)t} = e^{\Gamma t} + M_t \quad\text{and}\quad e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} = e^{\Gamma^{\mathrm{H}}t} + M_t^{\mathrm{H}}, \tag{12}$$

where M_t is an upper-triangular Nilpotent matrix. From (12), we have

$$\begin{aligned}\mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t}e^{(\Gamma+N)t}) &= \mathbf{Tr}(e^{\Gamma t}e^{\Gamma^{\mathrm{H}}t} + M_tM_t^{\mathrm{H}})\\ &\ge \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+\Gamma)t}).\end{aligned} \tag{13}$$

From (11) and (13), it follows that

$$\begin{aligned}\mathbf{Tr}(e^{A^{\mathrm{T}}t}e^{At}) &= \mathbf{Tr}(Ve^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t}e^{(\Gamma+N)t}V^{\mathrm{H}})\\ &\ge \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+\Gamma)t}) = \mathbf{Tr}(e^{2\mathbf{Re}\{\Gamma\}t}).\end{aligned} \tag{14}$$

Since $\mathbf{Re}\{\lambda_i(A)\} < 0$ for all $i = 1,\ldots,n$, we can conclude from (10) and (14) that

$$\mathbf{Tr}(P) = \int_0^\infty \mathbf{Tr}(e^{\bar{A}^{\mathrm{T}}t}e^{\bar{A}t})\,dt \ge \sum_{i=1}^{n} \frac{-1}{2\mathbf{Re}\{\lambda_i(A)\}}. \tag{15}$$

In the last inequality, we apply the fact that the trace and sum operators are linear and they can commute with the integral. The lower bound is achieved if and only if equalities in (14) and (13) hold, or equivalently, A is a normal matrix, i.e., $A^{\mathrm{T}}A = AA^{\mathrm{T}}$. In order to prove inequality (9), we first use Bernstein inequality (Bernstein, 1988)

$$\mathbf{Tr}(e^{A^{\mathrm{T}}t}e^{At}) \le \mathbf{Tr}(e^{(A^{\mathrm{T}}+A)t}). \tag{16}$$

Then, by taking an integral from both sides of (16) we get

$$\mathbf{Tr}(P) = \int_0^\infty \mathbf{Tr}(e^{A^{\mathrm{T}}t}e^{At})dt \le \int_0^\infty \mathbf{Tr}(e^{(A^{\mathrm{T}}+A)t})dt. \tag{17}$$

According to our assumptions $A + A^{\mathrm{T}}$ is Hurwitz. Therefore, using this fact and (17) we conclude that

$$\mathbf{Tr}(P) \le \int_0^\infty \mathbf{Tr}(e^{(A^{\mathrm{T}}+A)t})dt = \sum_{i=1}^n \frac{-1}{2\lambda_i(A_s)}. \quad \blacksquare$$

The following theorem shows that the lower and upper bounds in Theorem 2 can be tightened further by assuming more structure on the state matrix.

Theorem 3. *Suppose that linear system (2)–(3) is stable with input noise covariance (4) and normal state matrix A. Then,*

$$\sum_{i=1}^n \frac{-\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}} \le \boldsymbol{\rho_{ss}}(A; Q) \le \sum_{i=1}^n \frac{-\lambda_i(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}. \tag{18}$$

Moreover, both bounds in (18) are achieved if Q has n identical eigenvalues.

Proof. Every symmetric matrix Q can be decomposed as $Q = UDU^{\mathrm{T}}$, where $UU^{\mathrm{T}} = U^{\mathrm{T}}U = I$ and $D = \mathbf{diag}[\lambda_1(Q), \ldots \lambda_n(Q)]$. Thus, we can rewrite (7) as

$$\bar{A}^{\mathrm{T}}\bar{P} + \bar{P}\bar{A} + D = \mathbf{0}, \tag{19}$$

where $\bar{A} = U^{\mathrm{T}}AU$ and $\bar{P} = U^{\mathrm{T}}PU$. Since A is Hurwitz, the unique solution of (19) can be expressed by

$$\bar{P} = \int_0^\infty e^{\bar{A}^{\mathrm{T}}t}De^{\bar{A}t}dt. \tag{20}$$

Since $\bar{A}$ is normal, there exists a unitary matrix $\bar{V} \in \mathbb{C}^{n\times n}$ such that $\bar{A} = \bar{V}\Gamma\bar{V}^{\mathrm{H}}$, where $\Gamma = \mathbf{diag}[\lambda_1(A), \ldots, \lambda_n(A)]$ and $\bar{V}^{\mathrm{H}}$ is the conjugate transpose of $\bar{V}$. Next, let us consider the integrand of (20)

$$\begin{aligned}\mathbf{Tr}(e^{\bar{A}^{\mathrm{T}}t}De^{\bar{A}t}) &= \mathbf{Tr}(e^{\bar{A}^{\mathrm{H}}t}De^{\bar{A}t}) \\ &= \mathbf{Tr}(e^{\Gamma^{\mathrm{H}}t}\bar{V}^{\mathrm{H}}D\bar{V}e^{\Gamma t}\bar{V}^{\mathrm{H}}\bar{V}) \\ &= \mathbf{Tr}(\bar{V}^{\mathrm{H}}D\bar{V}e^{(\Gamma^{\mathrm{H}}+\Gamma)t}).\end{aligned} \tag{21}$$

We observe that $\bar{V}^{\mathrm{H}}D\bar{V}$ and $e^{(\Gamma^{\mathrm{H}}+\Gamma)t}$ are Hermitian. Thus, according to (Lasserre, 1995, Theorem. II. 2), we get

$$\mathbf{Tr}(\bar{V}^{\mathrm{H}}D\bar{V}e^{(\Gamma^{\mathrm{H}}+\Gamma)t}) \ge \sum_{i=1}^n \lambda_{n-i+1}(Q)e^{2\mathbf{Re}\{\lambda_i(A)\}t}. \tag{22}$$

Since $\mathbf{Re}\{\lambda_i(A)\} \ne 0$ for all $i = 1, \ldots, n$, from (20) and (22) we have

$$\begin{aligned}\mathbf{Tr}(P) &\ge \int_0^\infty \sum_{i=1}^n \lambda_{n-i+1}(Q)e^{2\mathbf{Re}\{\lambda_i(A)\}t}dt \\ &= -\sum_{i=1}^n \frac{\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}.\end{aligned} \tag{23}$$

In the last inequality, we apply the fact that the trace and sum operators are linear and they can be interchanged by the integral. When A is normal, then $\lambda_i(A + A^{\mathrm{T}}) = 2\mathbf{Re}\{\lambda_i(A)\}$. This is because according to the Schur decomposition for normal matrices, there exists a unitary $V \in \mathbb{C}^{n\times n}$ such that $A = V\Gamma V^{\mathrm{H}}$, where $\Gamma = \mathbf{diag}\{\lambda_1(A), \ldots (A), \lambda_n(A)\}$ and V^{H} denotes the conjugate transpose of matrix V. Using this fact, it follows that

$$\begin{aligned}A_s &= \frac{A + A^{\mathrm{H}}}{2} = V\left(\frac{\Gamma + \Gamma^{\mathrm{H}}}{2}\right)V^{\mathrm{H}} \\ &= V\mathbf{diag}\big(\mathbf{Re}\{\lambda_1(A)\}, \ldots, \mathbf{Re}\{\lambda_n(A)\}\big)V^{\mathrm{H}}.\end{aligned} \tag{24}$$

This implies that $\lambda_i(A_s) = \mathbf{Re}\{\lambda_i(A)\}$ for all $i = 1, \ldots, n$. In order to prove the RHS inequality in (18), we use (Komaroff, 1992, Corollary 2.1.1), which gives us the upper bound in (18). When Q has identical eigenvalues, then the upper and lower bounds have equal values; therefore, both bounds in (8) are achieved. $\blacksquare$

All symmetric and orthogonal matrices are examples of normal matrices. One of the outcomes of Theorem 3 is that when A is normal and Q has n identical eigenvalues, the value of the performance measure (5) is exactly equal to the upper and lower bounds in (18) and can be calculated as a function of eigenvalues of the symmetric part of A or equivalently the real parts of eigenvalues of A.

5. Applications to some network models

In this section, we apply the results of the previous section to some systems with specific interconnection topologies. One of the challenging problems in the area of linear dynamical networks is to discover relationships between the $\mathcal{H}_2$-norm of a linear network and the structure of its underlying interconnection topology. In general, carrying out such network analysis problems is difficult, if not impossible. In the following, we show that because of the particular functional form of bounds in Theorems 2 and 3, one can exploit structural properties of some classes of linear time-invariant networks in order to calculate their $\mathcal{H}_2$-norm bounds in more explicit forms and relate them to their graph topologies.

5.1. Linear consensus networks over directed graphs

We consider a class of linear consensus networks where the interconnection topology between subsystems is defined using a directed graph (Olfati-saber, Fax, & Murray, 2007; Siami & Motee, 2016). This class of networks can be modeled by (2)–(3) with $A = -L$, in which L is the Laplacian matrix of the underlying directed graph. We assume that all directed graphs in this section are weighted and strongly connected (Bang-Jensen & Gutin, 2008). As a result, we have $\lambda_1(L) = 0$ and $\mathbf{Re}\{\lambda_i(L)\} > 0$ for all $n = 2, \ldots, n$. In order to guarantee a well-defined and bounded $\mathcal{H}_2$-norm for this class of networks, it is further assumed that only stable modes of the network are observable from the output. We stress that in both Theorems 2 and 3, it is assumed that matrix A is Hurwitz. Next result extends Theorem 2 to marginally stable linear consensus networks over directed graphs.

Theorem 4. *Consider a linear system (2)–(3) with $A = -L$ and input noise covariance (4), where L corresponds to Laplacian matrix of a directed weighted graph that is strongly connected and balanced. Then, it follows that*

$$\sum_{i=2}^n \frac{1}{2\mathbf{Re}\{\lambda_i(L)\}} \le \boldsymbol{\rho_{ss}}(-L; Q) \le \sum_{i=2}^n \frac{1}{\lambda_i(L + L^{\mathrm{T}})}, \tag{25}$$

where $Q = M_n$ is the centering matrix. Moreover, the lower bound in (25) is achieved if and only if L is normal.

Proof. First, we show that if the underlying graph is balanced and strongly connected, then $L + L^{\mathrm{T}}$ has only one zero eigenvalue and the rest of them are strictly positive. Since the underlying graph is balanced, the row sum and column sum of Laplacian matrix L is zero. Therefore, $L + L^{\mathrm{T}}$ has zero row and column sums and it can be considered as Laplacian matrix of an undirected graph. However, this undirected graph is connected because L is the Laplacian matrix of a strongly connected graph. As a result, $L + L^{\mathrm{T}}$ has only one zero eigenvalue, i.e., $\lambda_1(L+L^{\mathrm{T}}) = 0$ and $\lambda_2(L+L^{\mathrm{T}}) > 0$. Now, let us define the disagreement vector by

$$x_d(t) := M_n x(t) = x(t) - \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}} x(t). \tag{26}$$

By multiplying a vector by the centering matrix, we actually subtract the mean of all the entries of the vector from each entry. The dynamics of linear network (2)–(3) with respect to the new state transformation (26) is so-called disagreement form of the network that is given by

$$\dot{x}_d(t) = -\left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right) x_d(t) + M_n\xi(t), \tag{27}$$

$$y(t) = x_d(t). \tag{28}$$

It can be shown that the transfer function of the network from ξ to y stays invariant under state transformation (26) (Siami & Motee, 2016). We show that (27)–(28) and the following system have identical performance measure (5):

$$\dot{x}_d(t) = -\left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)^{\mathrm{T}} x_d(t) + \xi(t), \tag{29}$$

$$y(t) = M_n x_d(t). \tag{30}$$

Both state matrices in (27) and (29) are Hurwitz. Therefore, the $\mathcal{H}_2$-norm of both systems from ξ to y are well-defined. The squared $\mathcal{H}_2$-norm of (27)–(28) is given by $\boldsymbol{\rho_{ss}}(A, Q) = \frac{1}{2}\mathbf{Tr}(P)$, where P is the unique solution of

$$P\left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right) + \left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)^{\mathrm{T}} P = M_n. \tag{31}$$

The squared $\mathcal{H}_2$-norm of (29)–(30) is given by

$$\lim_{t\to\infty} \mathbf{E}\left[y^{\mathrm{T}}(t)y(t)\right] = \frac{1}{2}\mathbf{Tr}(P_o), \tag{32}$$

where P_o is the unique solution of

$$\left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)^{\mathrm{T}} P_o + P_o\left(L + \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right) = M_n. \tag{33}$$

It is evident that both equations (31) and (33) return identical unique solutions, i.e., $P_o = P$. Hence, by applying Theorem 2 to system (29)–(30), we get the desired result. ■

The class of directed graphs with normal Laplacian matrices are balanced, but vice versa is not true in general.

Theorem 5. *Let us consider a linear consensus network over a strongly connected graph with Laplacian matrix L. If we assume that L is normal and $Q = C^{\mathrm{T}}C$ with $C\mathbf{1} = \mathbf{0}$, then it follows that*

$$\sum_{i=2}^{n} \frac{\lambda_i(Q)}{2\mathbf{Re}\{\lambda_i(L)\}} \le \boldsymbol{\rho_{ss}}(-L; Q) \le \sum_{i=2}^{n} \frac{\lambda_{n-i+2}(Q)}{2\mathbf{Re}\{\lambda_i(L)\}}. \tag{34}$$

Moreover, the lower and upper bounds in (34) are achieved if and only if $Q = q\left(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)$ for all $q \ge 0$.

The proof of Theorem 5 can be derived with some modifications from the proofs of Theorems 3 and 4. Similar to the proof of Theorem 4, first we need to form the disagreement network, and then, utilize Theorem 3 to conclude the proof.

Example 6. Let us consider a consensus network with a directed cycle graph given by Fig. 1, i.e., all the edges being oriented in the same direction with positive weight w. Without loss of generality, we may assume that $w = 1$. The Laplacian matrix of this graph is denoted by L_c which is a circulant matrix. According to results of

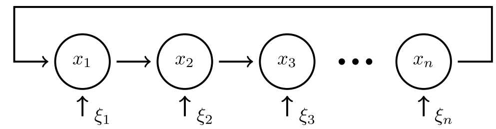

Fig. 1. Schematic diagram of a linear consensus network with a directed cycle graph with n agents. Each agent i is subject to stochastic disturbance ξ_i.

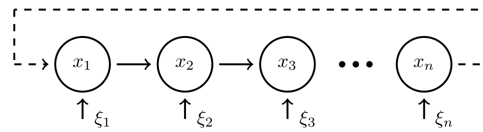

Fig. 2. Schematic diagram of a noisy cyclic network. The dashed link indicates a negative (inhibitory) feedback.

Norman (1973), the corresponding Laplacian eigenvalues are given by

$$\lambda_k(L_c) = 1 + \mathrm{e}^{\mathbf{i}\pi\left(1 - \frac{(-1)^k 2\left\lfloor \frac{k}{2}\right\rfloor}{n}\right)}, \tag{35}$$

where $k = 1, \dots, n$. As a result, their real parts can be calculated as

$$\mathbf{Re}\{\lambda_k(L_c)\} = 1 + \cos\left(\pi - \frac{(-1)^k 2\left\lfloor \frac{k}{2}\right\rfloor \pi}{n}\right) = 2\sin\left(\frac{2\pi\left\lfloor \frac{k}{2}\right\rfloor}{n}\right).$$

Since the corresponding underlying graph is strongly connected and its Laplacian matrix is normal, we can apply Theorem 5 to get

$$\sum_{k=2}^{n} \frac{\lambda_k(Q)}{2\sin\left(\frac{2\pi\left\lfloor \frac{k}{2}\right\rfloor}{n}\right)} \le \boldsymbol{\rho_{ss}}(-L_c; Q) \le \sum_{k=2}^{n} \frac{\lambda_{n-k+2}(Q)}{2\sin\left(\frac{2\pi\left\lfloor \frac{k}{2}\right\rfloor}{n}\right)}.$$

When $Q = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$, the performance measure can be calculated explicitly as a function of network size as $\boldsymbol{\rho_{ss}}(-L_c; Q) = \frac{n^2-1}{12}$. According to Definition 1, we conclude that the $\mathcal{H}_2$-norm of a linear consensus network with directed cycle graph deteriorates by $\mathcal{O}(n)$ as network size gets larger.

5.2. Linear networks with cyclic interconnection topology

The class of cyclic networks has been studied in the context of systems biology, e.g., in autocatalytic pathway with ring topology (Arcak & Sontag, 2007; Siami & Motee, 2013b; Siami, Motee, & Buzi, 2013; Tyson & Othmer, 1978). In order to obtain analytical bounds using our results from Section 4, we limit our attention to the class of linear cyclic networks shown in Fig. 2. We can represent the dynamics of the overall cyclic network in the compact canonical form (2)–(3) with the following state matrix

$$A = \begin{bmatrix} -a & 0 & \cdots & 0 & -c_n \\ c_1 & -a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -a & 0 \\ 0 & 0 & \cdots & c_{n-1} & -a \end{bmatrix}, \tag{36}$$

and output matrix $C = I_n$. In the next theorem, we use our results in Section 4 to exploit structural properties of this class of linear dynamical networks in order to compute their $\mathcal{H}_2$-norm bounds.

Theorem 7. *For the cyclic linear dynamical network with state matrix (36) and output matrix $C = I_n$, let us define $\mathfrak{c} := \sqrt[n]{c_1 c_2 \cdots c_n}$ and assume that the stability condition $\gamma := a/\mathfrak{c} > \cos(\pi/n)$ holds. Then the corresponding performance measure is lower bounded by*

$$\boldsymbol{\rho_{ss}}(A; Q) \ge \mathfrak{L}(n, \beta, \mathfrak{c}), \tag{37}$$

where

$$\mathfrak{L}(n,\beta,\mathfrak{c}) = \begin{cases} \dfrac{n\tan\frac{\beta}{2}}{2\mathfrak{c}\sin\frac{\beta}{n}} & \text{if} \quad \gamma < 1 \\ \dfrac{n^2}{4\mathfrak{c}} & \text{if} \quad \gamma = 1 \\ \dfrac{n\tanh\frac{\beta}{2}}{2\mathfrak{c}\sinh\frac{\beta}{n}} & \text{if} \quad \gamma > 1 \end{cases} \tag{38}$$

and

$$\beta := \begin{cases} \operatorname{arcos}(\gamma)n & \text{if} \quad \gamma \le 1 \\ \operatorname{arcosh}(\gamma)n & \text{if} \quad \gamma > 1. \end{cases} \tag{39}$$

The equality in (37) is achieved if and only if $c_1 = \cdots = c_n$, which means that all subsystems of the network are identical.

Proof. The stability condition $\gamma > \cos(\pi/n)$ implies that A is Hurwitz (Arcak & Sontag, 2007; Tyson & Othmer, 1978). Therefore, the $\mathcal{H}_2$-norm squared is well-defined and finite. The characteristic polynomial of A is given by

$$(\lambda + a)^n + c_1c_2\cdots c_n = 0. \tag{40}$$

Therefore, the eigenvalues of the matrix are

$$\lambda_k = -a + \mathfrak{c}e^{\mathbf{i}\left(\frac{\pi}{n}+\frac{2\pi k}{n}\right)} \tag{41}$$

for $k = 0, 1, \ldots, n-1$. By substituting these eigenvalues into the lower bound (9), we get

$$-\sum_{i=1}^{n}\frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} = \sum_{k=0}^{n-1}\frac{1}{2\mathbf{Re}\left\{-a + \mathfrak{c}e^{\mathbf{i}\left(\frac{\pi}{n}+\frac{2\pi k}{n}\right)}\right\}} = \sum_{k=0}^{n-1}\frac{1}{2\mathfrak{c}\left(\gamma - \cos\left(\frac{\pi}{n}+\frac{2\pi k}{n}\right)\right)}. \tag{42}$$

First, let us assume that $\gamma < 1$ and substitute $\gamma = \cos(\beta/n)$ in (42). It follows that

$$\begin{aligned} -\sum_{i=1}^{n}\frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} &= \frac{1}{2\mathfrak{c}}\sum_{k=0}^{n-1}\frac{1}{\cos\left(\frac{\beta}{n}\right) - \cos\left(\frac{\pi}{n}+\frac{2\pi k}{n}\right)} \\ &= \frac{1}{4\mathfrak{c}}\sum_{k=0}^{n-1}\csc\left(\frac{(2k+1)\pi}{2n}+\frac{\beta}{2n}\right) \\ &\quad \times \csc\left(\frac{(2k+1)\pi}{2n}-\frac{\beta}{2n}\right) \\ &= \frac{n\tan\frac{\beta}{2}}{2\mathfrak{c}\sin\frac{\beta}{n}}, \end{aligned}$$

where the Birkhoff Ergodic theorem is used to conclude the last equation. Similar steps can be taken when $\gamma > 1$. In each case by substituting γ from (39) in (42), one can obtain the desired result (37). According to Theorem 2, the equality in (37) is achieved if and only if A is a normal matrix. On the other hand, based on the cyclic structure of matrix (36), we conclude that A is normal if and only if $c_1 = \cdots = c_n$. ■

The secant criterion reported in Arcak and Sontag (2007) and Tyson and Othmer (1978) for cyclic linear networks provides a stability condition. This condition implies that the unperturbed system with $\xi = 0$ is stable if and only if $\gamma > \cos(\pi/n)$. For a fixed parameter β, the stability condition of the cyclic network is not affected when the number of intermediate subsystems changes. However, the result of Theorem 7 asserts that the lower bound of the performance measure (5) increases (i.e., the network performance deteriorates) when the network size increases. More explicitly, we have the following approximation for the lower bound (38)

$$\mathfrak{L}(n,\beta,\mathfrak{c}) \approx \begin{cases} \dfrac{\tan\frac{\beta}{2}}{2\mathfrak{c}\beta}n^2 & \text{if} \quad \gamma < 1 \\ \dfrac{1}{4\mathfrak{c}}n^2 & \text{if} \quad \gamma = 1 \\ \dfrac{\tanh\frac{\beta}{2}}{2\mathfrak{c}\beta}n^2 & \text{if} \quad \gamma > 1. \end{cases} \tag{43}$$

According to Definition 1, we conclude that when parameter β is fixed, the $\mathcal{H}_2$-norm of the cyclic network deteriorates in the order of $\Omega(n)$ as the network size becomes larger. We should mention that the $\mathcal{H}_2$-norm of n-identical coupled subsystems may scale in different orders depending on their underlying graph topology, for more details please see Siami and Motee (2016).

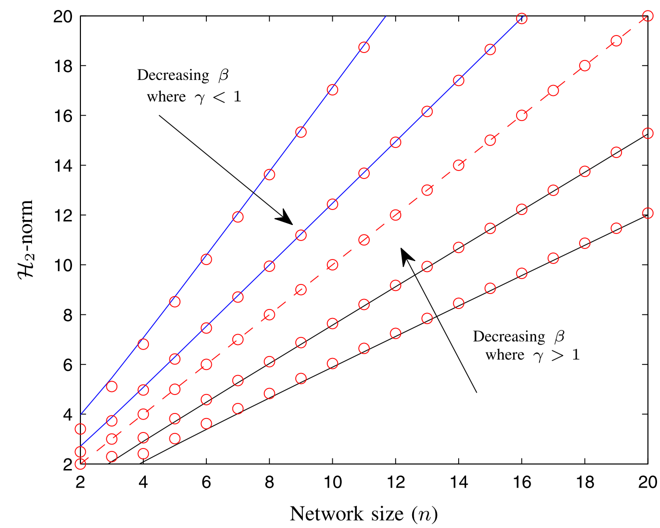

Fig. 3. The lower bound (38), which is depicted by small red circles (○), is compared asymptotically to its approximation in (43). It can be observed that (43) tightly approximates (38). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 8. In order to support our theoretical results, we consider a cyclic network (2)–(3) with state matrix (36), $\mathfrak{c} := c_1 = \cdots = c_n$, and $C = I_n$. The asymptotic scaling of the $\mathcal{H}_2$-norm for this class of networks is depicted in terms of network size and parameter β in Fig. 3. In this case, the $\mathcal{H}_2$-norm of the cyclic network can be calculated by the square root of (38). These values are depicted by small red circles (○) versus the number of subsystems n. Moreover, these values are compared asymptotically to their approximation given by the square root of (43). It can be observed that the square root of (43) tightly approximates the $\mathcal{H}_2$-norm of the cyclic network.

6. Tightness of our new bounds

In this section, we compare our results with the existing results in the literature. In Table 1, we summarize several lower bounds on the $\mathcal{H}_2$-norm squared of system (2)–(3) based on all existing works in the literature to the best of our knowledge. When $Q = I_n$, the lower bound in Theorem 2 is tighter than all existing lower bounds reported in reference papers (Komaroff, 1988; Kwon et al., 1985; Wang et al., 1986). In the following, we provide analytical proofs

Table 1
Comparison of existing lower bounds on ρ_{ss} in the literature.

Methods	Lower bounds
Theorem 2	$\sum_{i=1}^{n} \frac{-1}{2\mathbf{Re}\{\lambda_i(A)\}}$
Theorem 3	$\sum_{i=1}^{n} \frac{-\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}$
Mori et al. (1987)	$-\frac{\mathbf{Tr}(Q)}{2\lambda_{\min}(A_s)}$
Kwon et al. (1985)	$-\frac{n^2\lambda_{\min}(Q)}{2\mathbf{Tr}(A)}$
Wang et al. (1986)	$-\frac{\mathbf{Tr}(Q)}{2\mathbf{Tr}(A)}$
Komaroff (1988)	$-\frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{2\mathbf{Tr}(A)}$
Lee (1997)	$\sum_{i=1}^{n}\lambda_i\left(\frac{Q}{a}-\frac{AA^{\mathrm{T}}}{a^2}\right)^{\frac{1}{2}}$ for $Q \succ \frac{AA^{\mathrm{T}}}{a}$

for our claim. It is true that

$$-\mathbf{Tr}(A) = -\sum_{i=1}^{n}\mathbf{Re}\{\lambda_i(A)\}, \tag{44}$$

and $-\mathbf{Re}\{\lambda_i(A)\}$ are positive for all $i = 1,\ldots,n$. From the arithmetic and harmonic mean inequalities, it follows that

$$\frac{-n^2}{2\mathbf{Tr}(A)} \le \sum_{i=1}^{n}\frac{-1}{2\mathbf{Re}\{\lambda_i(A)\}}. \tag{45}$$

Moreover, we have that $\frac{-n}{2\mathbf{Tr}(A)} \le \frac{-n^2}{2\mathbf{Tr}(A)}$. As a result, when $Q = I_n$ the following ordering on bounds holds

$$\underbrace{\frac{-n}{2\mathbf{Tr}(A)}}_{\text{(Wang et al., 1986)}} \le \underbrace{\frac{-n^2}{2\mathbf{Tr}(A)}}_{\text{(Komaroff, 1988; Kwon et al., 1985)}} \le \underbrace{\sum_{i=1}^{n}\frac{-1}{2\mathbf{Re}\{\lambda_i(A)\}}}_{\text{Theorem 2}}.$$

On the other hand, if A is normal, then the lower bound in Theorem 3 is tighter than the lower bounds presented in Komaroff (1988), Kwon et al. (1985) and Wang et al. (1986). In the next few lines, we will prove this claim. We know that

$$\frac{\mathbf{Tr}(Q)}{-2\mathbf{Tr}(A)} = \frac{\sum_{i=1}^{n}\lambda_i(Q)}{-2\mathbf{Tr}(A)} \le \frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{-2\mathbf{Tr}(A)} \tag{46}$$

and

$$\frac{\mathbf{Tr}(Q)}{-2\mathbf{Tr}(A)} \le \frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{-2\mathbf{Tr}(A)}. \tag{47}$$

From the Cauchy–Schwarz inequality, we get

$$\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2 \le \sum_{i=1}^{n}\frac{\lambda_{n-i+1}(Q)}{-\mathbf{Re}\{\lambda_i(A)\}}\left(-\sum_{i=1}^{n}\mathbf{Re}\{\lambda_i(A)\}\right). \tag{48}$$

Then, Eq. (44) and inequality (48) give us

$$-\frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{2\mathbf{Tr}(A)} \le -\sum_{i=1}^{n}\frac{\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}. \tag{49}$$

According to (46), (47) and (49), we conclude that our proposed lower bound is tighter than the lower bounds reported in Komaroff (1988), Kwon et al. (1985) and Wang et al. (1986). In summary, we have

$$\underbrace{\frac{\mathbf{Tr}(Q)}{-2\mathbf{Tr}(A)}}_{\text{(Wang et al., 1986)}} \le \underbrace{\frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{-2\mathbf{Tr}(A)}}_{\text{(Komaroff, 1988)}} \le \underbrace{\sum_{i=1}^{n}\frac{-\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}}_{\text{Theorem 3}}$$

and

$$\underbrace{\frac{n^2\lambda_{\min}(Q)}{-2\mathbf{Tr}(A)}}_{\text{(Kwon et al., 1985)}} \le \underbrace{\frac{\left(\sum_{i=1}^{n}\lambda_i(Q)^{\frac{1}{2}}\right)^2}{-2\mathbf{Tr}(A)}}_{\text{(Komaroff, 1988)}} \le \underbrace{\sum_{i=1}^{n}\frac{-\lambda_{n-i+1}(Q)}{2\mathbf{Re}\{\lambda_i(A)\}}}_{\text{Theorem 3}}.$$

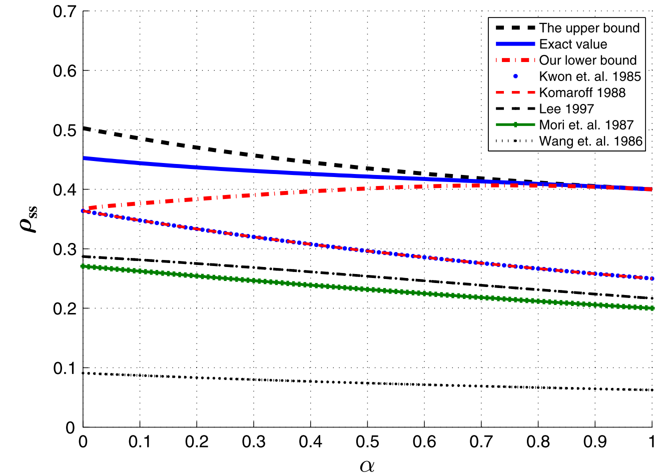

Fig. 4. A numerical comparison of the results presented in Table 1 for the family of linear systems given in Example 9.

To support our results, we illustrate by means of two simulation examples that our lower bounds for the performance measure (5) are the tightest among the other known bounds given in Table 1.

Example 9. Let us define the parametrized family of matrices A_α as follows

$$A_\alpha = (1-\alpha)A_0 + \alpha A_1 \tag{50}$$

for all $0 \le \alpha \le 1$, where A_0 and A_1 are given by

$$A_0 = \begin{bmatrix} -5 & 3 & 3 & 3 \\ 0 & -5 & 2 & 2 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & -6 \end{bmatrix} \quad \text{and}$$

$$A_1 = \begin{bmatrix} -8 & 3 & 2 & 1 \\ 1 & -8 & 3 & 2 \\ 2 & 1 & -8 & 3 \\ 3 & 2 & 1 & -8 \end{bmatrix}.$$

We evaluate the performance of the parametrized family of linear system (2)–(3) with state matrix A_α and output matrix $C = I_4$ for all $0 \le \alpha \le 1$. In Fig. 4, our lower bound based on Theorem 2 is compared with other known bounds summarized in Table 1. One observes from this figure that our lower bound outperforms all existing lower bounds for all $0 \le \alpha \le 1$. For all $0 \le \alpha < 1$, the parametrized matrix A_α is not normal. However, this matrix becomes normal for $\alpha = 1$. Therefore, as it is seen in the figure our lower bounds reach the exact value of the performance measure for $\alpha = 1$.

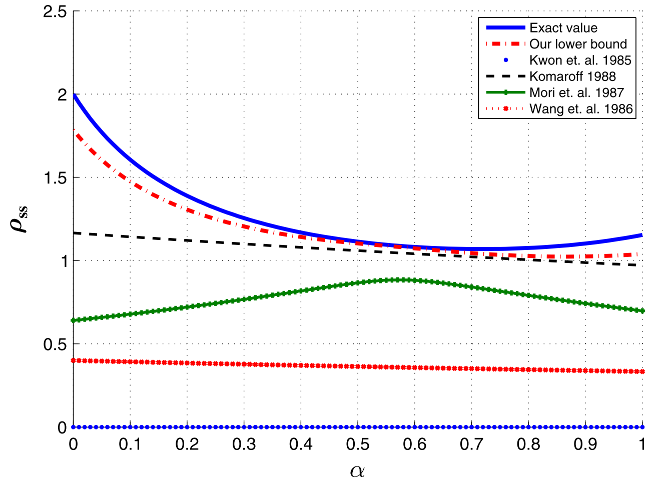

Fig. 5. A numerical comparison of the results presented in Table 1 for the family of consensus networks given in Example 10.

Example 10. We illustrate tightness of our bounds on the performance measure of the parametrized family of linear consensus networks over directed graph that are defined using the following parametrized family of Laplacian matrices

$$L_\alpha = (1-\alpha)L_0 + \alpha L_1 \tag{51}$$

for all $0 \le \alpha \le 1$, where L_0 and L_1 are given by

$$L_0 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ 0 & -1 & -2 & 3 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} 4 & -2 & 0 & -2 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -2 & 0 & -1 & 3 \end{bmatrix}$$

and the following output matrix

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

In Fig. 5, our lower bound is compared with other known bounds presented in Table 1. The simulation results confirm that our proposed lower bound is tighter than all the previously existing results in the literature. It should be noted that since $Q = C^{\mathrm{T}}C$ is not full-rank (i.e., singular) the result of Lee (1997) is not applicable to this family of systems.

7. Discussion and conclusion

The proposed lower and upper bounds in Theorem 2 are functions of the real parts of eigenvalues of the state matrix of the system and the eigenvalues of its symmetric part, respectively. We have shown that the spectral lower bound in Theorem 2 is tighter than all existing lower bounds reported in Table 1. Our proposed lower bound requires computation of all eigenvalues of a $n \times n$ state matrix that in general has higher computational complexity than those lower bounds in Table 1. This extra complexity is the price of having comparably tighter estimates for the performance measure. Calculation of eigenvalues (modes) of some classes of linear dynamical networks with normal or symmetric state matrices are inexpensive and may lead to closed-form expressions for all eigenvalues, e.g., spatially invariant systems with lattice or ring topologies and linear consensus networks with path, star, cycle, complete, and complete bipartite graph topologies; see Bamieh et al. (2012) and Siami and Motee (2016) and references in there. For networks with generic $n \times n$ state matrices, the best currently known bounds for arithmetic complexity of computing all eigenvalues and their associated eigenspaces is given by $\mathcal{O}\big(n^3 + (n\log^2 n)\log b\big)$ for an approximation within 2^{-b}; see Pan and Chen (1999) for more details. This bound is reported to be optimal up to a logarithmic factor, where it is shown that a much better randomized arithmetic complexity of order $\mathcal{O}\big(n^2\log n + (n\log^2 n)\log b\big)$ can be achieved for some important special classes of matrices such as Toeplitz, Hankel, Toeplitz-like, Hankel-like, and Toeplitz-like-plus-Hankel-like matrices.

The value of having a spectral lower bound like (8) is beyond its computational complexity as it provides valuable insight on how the expected output energy under white noise excitation depends on the dynamic modes of the system, which is given by the lower bound $\frac{-1}{2}\sum_{i=1}^{n}\mathbf{Re}\{\lambda_i\}^{-1}$. We may think of term $\frac{-1}{2}\mathbf{Re}\{\lambda_i\}^{-1}$ as a quantity that can be associated with the energy of the i'th mode of the system, which is inversely proportional to its distance from the imaginary axis in the complex domain. The first key point is that for networks with a few slow modes, we can still obtain rather tight lower bounds by only identifying those slow modes; for example see Ljung (1998) for some efficient identification algorithms. The second key point about the spectral lower bound is that it helps to unravel the fundamental role of slow modes in performance deterioration: slower modes are more energetic and dominant after transient phase in time, i.e., the high energy components of the output signal are the ones that are temporally slow. This suggests some useful insights on how to design inter-network feedback control laws by replacing slow modes of the network in order to achieve better performance bounds. This is one of our future research directions.

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